Table of Contents

Foreword to First Edition

This book contains an exposition of the results of William Thurston in the theory of surfaces (measured foliations, natural compactification of Teichmüller space, and classification of surface diffeomorphisms). Our scope is essentially that outlined in the research announcements of Thurston, and in the notes of his Princeton course, as written up by M. Handel and W. Floyd.

A part of this work, notably the classification of curves and of measured foliations, is an elaboration of expositions made in the Seminaire d'Orsay in 1976–1977. But we were not able to write the proofs for the remaining portions of the theory until much later. In the Spring of 1978, at Plans-Sur-Bex, Thurston explained to us how to see the projectification of the space of measured foliations as the boundary of Teichmüller space.

The first exposé enumerates the principal results, the proofs of which follow in exposés 2 through 13. The last two exposés present work somewhat marginal to the theme of the classification of surface diffeomorphisms. Exposé 14 (orally presented by D. Fried and D. Sullivan) discusses nonsingular closed 1-forms on 3-dimensional manifolds, following Thurston, in particular on the fibres on S^1 for which the monodromy diffeomorphism is pseudo-Anosov. Exposé 15 (orally given by A. Marin) gives a finite presentation of the mapping class group following Hatcher and Thurston.

The seminar consisted also of exposés of an analytical nature (holomorphic quadratic differentials, quasi-conformal mappings) given by W. Abikoff, L. Bers, and J. Hubbard. In the end, the two points of view were found to be more independent of each other than was initially believed. The analytic point of view is the subject of a separate text written by W. Abikoff (see [Abi72]).

We thank all of the active participants in the seminar; all have contributed assistance in various sections. A. Douady, who, after the oral presentations, helped us to capture the content of the lectures; M. Shub, who discussed with us the ergodic point of view; D. Sullivan, who besides giving much advice and encouragement, strove to make us understand how the image of a curve under iteration of a pseudo-Anosov diffeomorphism "finishes" the foliation of the surface. (It took many more months to fully understand this "mixing".)

Finally, we thank Mme. B. Barbichou and S. Berberi for the care which

they took in typing the manuscript and producing the illustrations.

Translator's notes

I would like to thank Dale Rolfsen, for providing support, encouragement and many fruitful discussions, the Mathematics departments at Berkeley and McGill, for providing library and computer facilities during visits in 1990 and 1996 respectively, and Elisabeth Kim, who read early drafts of some of the chapters and provided corrections and helpful suggestions.

This book was typeset using a modified LaTeX 2ϵ report style. Most figures were produced using `xfig`, free software written by Supoj Sutanthavibul, Ken Yap, Brian Smith, and Micah Beck, among others. The resulting figures were converted into encapsulated PostScript using `transfig` and then further post-processed. The drafts were printed using `dvips`.

Thanks to: Mary Ann Lacey, Ronald Ferguson, Kevin Pilgrim, Jonathan Walden, Frances Fry, Bill Casselman, Curt McMullen.

Djun Kim,
Vancouver, B.C.
djun@math.ubc.ca

Abstract

This book is a an exposition of Thurston's theory of surfaces (measured foliations, compactification of Teichmüller space and classifications of diffeomorphisms).

The mathematical content is roughly the following.

For a surface M (let's say closed, orientable, of genus $g > 1$), one defines $\mathcal{S}$ as the set of isotopy classes of simple closed curves in M. For $\alpha, \beta \in \mathcal{S}$, one denotes by $i(\alpha, \beta)$ the minimum number of *geometric* intersection points of α' with β', where α' (resp. β') is a simple curve in the class α (resp. β). This induces a map $i_*\colon \mathcal{S} \xrightarrow{\mathbb{R}} {}_+^{\mathcal{S}}$ which turns out to be injective. In fact, if one projectivizes $\mathbb{R}_+^{\mathcal{S}} \setminus 0$, i_* induces an injection $i_*\colon \mathcal{S} \xrightarrow{P} (\mathbb{R}_+^{\mathcal{S}})$ which endows $\mathcal{S}$ with a nontrivial topology. Here $\mathbb{R}_+^{\mathcal{S}}$ is provided with the weak topology (= product topology). Two curves $\alpha, \beta \in \mathcal{S}$ are "near to each other" in $P(\mathbb{R}_+^{\mathcal{S}})$ if, up to a multiple, they are made up of more or less the same strands going more or less the same direction. This has nothing to do with homotopy theory.

The limits of curves are naturally interpreted as projective classes of "measured foliations", which means foliations with an "invariant" transverse distance, having a certain kinds of singularities (well-known in the theory of quadratic differentials, or in smectic liquid crystals). The space of measured foliations considered in $\mathbb{R}_+^{\mathcal{S}}$ (or in $P(\mathbb{R}_+^{\mathcal{S}})$) is denoted by $\mathcal{MF}$ (resp. $P\mathcal{MF}$). One shows that:

$$\mathcal{MF} \simeq \mathbb{R}^{6g-6}. \qquad P\mathcal{MF} \simeq S^{6g-7}.$$

In $P(\mathbb{R}_+^{\mathcal{S}})$, $P\mathcal{MF}$ and the Teichmüller space $\mathcal{T}(M)$ glue together into a $6g - 6$ dimensional disk:

$$\bar{\mathcal{T}}(M) = \mathcal{T}(M) \cup P\mathcal{MF}(M) = D^{6g-6}.$$

The group $\mathrm{Diff}(M)$ acts continuously on this compactification of $\mathcal{T}$ (this is hence "a natural compactification").

Hence any $\phi \in \mathrm{Diff}(M)$ has a fixed point in $\bar{\mathcal{T}}(M)$ (Brouwer) and the analysis of this fixed point shows that (up to isotopy) each ϕ is either a hyperbolic isometry, or "Anosov-like" (the word is "pseudo-Anosov"), or else "reducible".

Pseudo-Anosov diffeomorphisms minimize the topological entropy in their isotopy class. Also two pseudo-Anosov's which are isotopic are actually conjugate.

Every diffeomorphism $\phi\colon M \to M$ has a (finite) spectrum defined in terms of the length of $\phi^n\alpha'$ raised to the power $1/n$. A pseudo-Anosov is characterized by the fact that the spectrum is reduced to a single value $\lambda > 1$.

There is a good method to produce many pseudo-Anosov's out of combinations of Dehn twists which is explained in exposé 13.

The last two chapters are of a somewhat different character: exposé 14 is about closed non-singular 1-forms on 3-manifolds, and exposé 15 about the Hatcher-Thurston theorem on finite presentability of $\pi_0\mathrm{Diff}(M)$.

Chapter 1

Collected theorems of Thurston on Surfaces

by V. Poénaru

1.1 Introduction

Thurston's theory ([Thu67], see also [Thu], [Poen78]) is concerned with the following three questions:

I. Describe "all" closed curves without a double point (not necessarily connected) on a surface, up to isotopy.

II. Describe "all" diffeomorphisms of a surface, up to isotopy.

III. Give a natural (with respect to the action of diffeomorphisms) boundary for Teichmüller space.

For a closed surface, there always exists a Riemannian metric of constant curvature [Gau27]. The table below (Table 1.1) summarizes the possibilities and at the same time establishes a parallel between the geometric and the topological properties.

Most of Thurston's theorems hold for any compact surface, but in the following, we restrict ourselves to orientable surfaces which are closed, or which have non-empty boundary.

1.2 The space $\mathcal{S}$ of curves

Let M be a compact, connected, orientable surface. We write $\mathcal{S}(M) = \mathcal{S}$ for the set of isotopy classes of simple, closed, connected curves of M which are not homotopic to zero or homotopic to a boundary component of M.

Surface:	K **(curvature):**	χ **(Euler characteristic):**	**Remarks:**
S^2, $\mathbb{R}P^2$	$K = 1$ (Elliptic geometry)	$\chi > 0$	π_1 is finite, $\pi_2 \neq 0$.
T^2, K^2 (Klein bottle)	$K = 0$ (Euclidean geometry)	$\chi = 0$	These are $K(\pi, 1)$'s and their universal covering space is $\mathbb{R}^2$.
genus > 1	$K = -1$ (Hyperbolic geometry)	$\chi < 0$	

Table 1.1: The three possible geometries on surfaces.

Remarks.

(1) The elements of $\mathcal{S}$ are *not* oriented.

(2) Since two simple closed curves which are homotopic are also isotopic [Eps66a], we may replace "isotopy classes" in the above definition with "homotopy classes".

Consider the symmetric map

$$i\colon \mathcal{S} \times \mathcal{S} \to \mathbb{Z}^+ = \{0, 1, 2, \dots\}$$

geometric intersection number

algebraic intersection number

defined in the following fashion: $i(\alpha, \beta)$ is the minimum number of intersections of a representative for α with a representative for β. This is the **geometric intersection number** (as opposed to the **algebraic intersection number**).

Example. On the torus T^2, we choose two oriented generators x and y. Then all elements of $\mathcal{S}$ may be represented by $\gamma(a, b) = ax + by$, where $a, b \in \mathbb{Z}$ and $(a, b) = 1$; in $\mathcal{S}$, we have $\gamma(a, b) = \gamma(-a, -b)$. The following formula is easy to verify:

$$i\left(\gamma(a, b), \gamma(c, d)\right) = \left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

Lemma 1.1

(1.) If $\alpha \in \mathcal{S}$, there is a $\beta \in \mathcal{S}$ such that $i(\alpha, \beta) \neq 0$.
(2.) If $\alpha_1 \neq \alpha_2$ in $\mathcal{S}$, there is a $\beta \in \mathcal{S}$ such that $i(\alpha_1, \beta) = 0 \neq i(\alpha_2, \beta)$.

The proof is given in chapter 2.3.

1.2.1 The space of functionals.

We may consider the set $\mathbb{R}_+^{\mathcal{S}}$ of functions from $\mathcal{S}$ to the non-negative reals, with the weak topology. The usual multiplication by the positive reals defines **rays** in $\mathbb{R}_+^{\mathcal{S}}$. The set of these is the projective space $P(\mathbb{R}_+^{\mathcal{S}})$; it is given the quotient topology. We have the natural mappings rays

$$\mathcal{S} \xrightarrow{i_*} \mathbb{R}_+^{\mathcal{S}} \setminus 0 \xrightarrow{\pi} P(\mathbb{R}_+^{\mathcal{S}})$$

where i_* is defined by $i_*(\alpha)(\beta) = i(\alpha, \beta)$. By (1) of lemma 1.1, $i_*(\mathcal{S})$ does not contain 0; (2) assures the injectivity of $\pi \circ i_*$.

Consider the completion of $\mathcal{S}$, denoted $\overline{\mathcal{S}}$, which is the closure of $\pi \circ i_*(\mathcal{S})$ in $P(\mathbb{R}_+^{\mathcal{S}})$. The elements of $\overline{\mathcal{S}}$ are represented by sequences $\{(t_n, \alpha_n)\}$, $t_n > 0$, $\alpha_n \in \mathcal{S}$, such that for all β in $\mathcal{S}$, the sequence of real numbers $t_n i(\alpha_n, \beta)$ converges.

Thus, within $P(\mathbb{R}_+^{\mathcal{S}})$, the set $\mathcal{S}$ is topologically non-trivial. Intuitively, we may give a meaning to the notion "two curves γ, γ' are close to each other". This 'proximity' has nothing to do with the respective homotopy classes of the curves, but with the fact that, up to a multiple, in every region of the surface, γ and γ' are more-or-less made up of the same number of strands, going in more-or-less the same direction. All of this will be discussed in greater detail in Chapter 3.4.

We need to introduce also the space $\mathcal{S}'$ of isotopy classes of simple, closed, not necessarily connected curves on M, whose every component "belongs" to $\mathcal{S}$. But two distinct components of the same curve are allowed to be isotopic to each other, so that we may consider a scalar multiplication: for an integer $n > 0$ and $\gamma \in \mathcal{S}'$, $n\gamma$ is represented by n parallel curves.

As before, we define $i\colon \mathcal{S}' \times \mathcal{S} \to \mathbb{Z}_+$, and obtain the diagram

$$\mathcal{S}' \xrightarrow{i_*} \mathbb{R}_+^{\mathcal{S}} \setminus 0 \xrightarrow{\pi} P(\mathbb{R}_+^{\mathcal{S}})$$

Clearly, i_* respects multiplication by scalars, hence πi_* is not injective on $\mathcal{S}'$. But one may easily show that πi_* admits $\overline{\mathcal{S}}$ as closure (see Chapter 3.4). In the following, we denote by $\mathbb{R}_+ \times \mathcal{S}$ the cone on $i_*(\mathcal{S})$ in $\mathbb{R}_+^{\mathcal{S}}$.

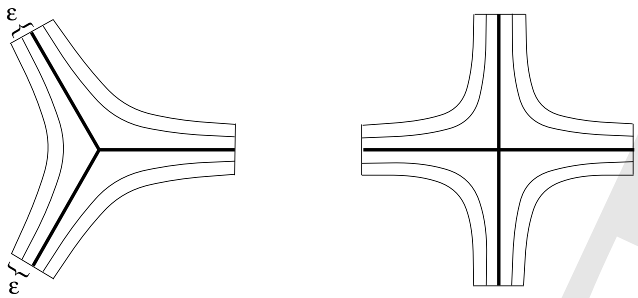


Figure 1.1: p-saddles, for $p = 3$, $p = 4$.

Theorem 1.2 *If M is a closed orientable surface of genus $g > 1$, then $\overline{\mathcal{S}}$ is homeomorphic to S^{6g-7}. (This is proved in chapter 3.4.)*

Let $M^2_{g,b} := \underset{g}{\#}(S^1 \times S^1) - \bigcup_b D^2$. If $\chi(M^2_{g,b}) < 0$, then $\overline{\mathcal{S}(M^2_{g,b})}$ is homeomorphic with $S^{6g+2b-7}$ (see Chapter 10.6). Lastly, $\overline{\mathcal{S}(T^2)} \simeq S^1$, $\overline{\mathcal{S}(D^2)} = \overline{\mathcal{S}(S^2)} = \overline{S^1 \times [0,1]} = \emptyset$.

1.3 Measured Foliations

measured foliation

For simplicity, M will be closed. A **measured foliation** on M is a foliation $\mathcal{F}$ with singularities (of the type of a holomorphic quadratic differential $z^{p-2}dz^2, p = 3, 4, \ldots$) together with a transverse measure invariant under holonomy. In the neighbourhood of a non-singular point, there exists a chart $\phi\colon U \to \mathbb{R}^2_{x,y}$ such that $\phi^{-1}(y = \text{constant})$ consists of the leaves of $\mathcal{F}|U$. If $U_i \cap U_j$ is non-empty, there exist transition functions ϕ_{ij} of the form

$$\phi_{ij}(x, y) = (h_{ij}(x, y), c_{ij} \pm y)$$

where c_{ij} is a constant. In this chart, the transverse measure is given by $|dy|$.

Remark. The foliations which admit transition functions of the form $(f(x, y), c+y)$ are those which are defined by a closed 1-form ω; away from singularities, y is a local primitive for ω.

The singularities of $\mathcal{F}$ are p-saddles ($p \geq 3$) as in figure 1.1.

total variation

If γ is a simple closed curve on M, we call $\int_\gamma \mathcal{F}$ the **total variation** of the

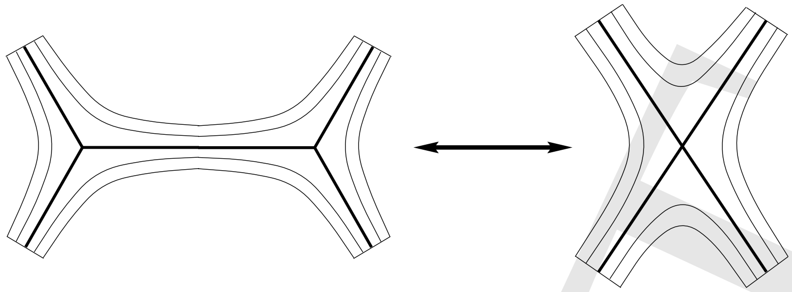


Figure 1.2:

coordinate y of $p \in \gamma$ as p traverses γ. For $\alpha \in \mathcal{S}$, define

$$I(\mathcal{F}, \alpha) := \inf_{\gamma \in \alpha} \int_\gamma \mathcal{F}.$$

One says that $\mathcal{F}_1$ and $\mathcal{F}_2$ are **equivalent in the sense of Whitehead** if the one may be transformed to the other by isotopies and elementary deformations of the type suggested by figure 1.2.

equivalent in the sense of Whitehead

(Observe that these deformations permit the transport of the transverse measure.) Denote by $\mathcal{MF}$ the set of equivalence classes. Define

$$I_*\colon \mathcal{MF} \to \mathbb{R}_+^{\mathcal{S}}$$

by

$$I_*(\mathcal{F})(\alpha) = I(\mathcal{F}, \alpha).$$

One says that $\mathcal{F}_1$ and $\mathcal{F}_2$ are **m-equivalent** (or **equivalent in the sense of Schwartz**) if $I_*(\mathcal{F}_1) = I_*(\mathcal{F}_2)$. Schwartz equivalence is an immediate consequence of Whitehead equivalence.

m-equivalent

equivalent in the sense of Schwartz

Theorem 1.3 *The map I_* injects $\mathcal{MF}$ into $\mathbb{R}_+^{\mathcal{S}}$; $I_*(\mathcal{MF}) \cup 0 = \overline{\mathbb{R}_+ \times \mathcal{S}}$, and if $g > 1$, this set is homeomorphic with $\mathbb{R}^{6g-g}$. In particular, Schwartz equivalence is the same thing as Whitehead equivalence.*

The proof of this theorem is dealt with in chapters 4.3 and 5.3.2. On the other hand, since $I_*(\mathcal{MF})$ misses 0, the theorem says that in $P(\mathbb{R}_+^{\mathcal{S}})$ we have $\overline{\mathcal{S}} = \pi \circ I_*(\mathcal{MF})$. This gives a nice geometric representation of the functionals in $\mathbb{R}_+ \times \mathcal{S}$.

1.4 Teichmüller Space

We will consider a surface M with $\chi(M) < 0$. Consider the space $\mathcal{H}$ of all metrics on M with constant curvature $K = -1$, such that every component of the boundary of M is a geodesic. Let $\mathrm{Diff}^0(M)$ be the group of diffeomorphisms isotopic to the identity, with the C^∞ topology. As we shall see later, this acts freely and continuously on $\mathcal{H}$. The orbit space under this action, equipped with the quotient topology, we will call the **Teichmüller space** $\mathcal{T}(M) = \mathcal{T}$. If M is orientable, there is another definition in terms of complex structures on M. The equivalence of the two definitions is a consequence of the uniformization theorem [Wey55].

Teich müller space

Remarks. Consider a fixed M, together with another surface $X_\rho = X$ with a hyperbolic metric ρ. If $\phi\colon M \to X$ is a diffeomorphism, the pair (X, ϕ) is called a **Teichmüller surface**.

Teich müller surface equivalent

Two Teichmüller surfaces (X, ϕ), (X', ϕ') are said to be **equivalent** if there is an isometry $f\colon X \to X'$ such that ϕ and $f \circ \phi'$ are isotopic.

It is convenient to identify the points of $\mathcal{T}$ with equivalence classes of Teichmüller surfaces.

We remark here that two diffeomorphisms of M are homotopic if and only if they are isotopic (see [Eps66a]).

If M is closed, of genus $g > 1$, a classical theorem of Teichmüller theory asserts that

$$\mathcal{T}(M) \simeq \mathbb{R}^{6g-6}.$$

This result, due to Fricke and Klein, will be re-proved in chapter 6.

Further, we have

$$\mathcal{T}(M^2_{g,b}) \simeq \mathbb{R}^{6g-6+2b}.$$

For all $\theta \in \mathcal{T}$ and $\alpha \in \mathcal{S}$, we define

$$\ell(\theta, \alpha) := \inf_{\gamma \in \alpha} (\theta(\gamma))$$

where $\theta(\gamma)$ designates the length of γ computed in the metric prescribed up to isotopy on M.

The metric being fixed, the infimum is attained for a unique geodesic. From the above formula, we obtain the map

$$\ell_*\colon \mathcal{T} \to \mathbb{R}_+^{\mathcal{S}}$$

It can be easily seen that the image of the map misses $I_*(\mathcal{MF}) \cup 0$. The group $\pi_0(\mathrm{Diff}(M))$ acts on Teichmüller space as it does on $\mathcal{S}$, thus on $\mathbb{R}_+^{\mathcal{S}}$; the map ℓ_* is clearly equivariant.

In chapter 6, we prove the theorem below.

Theorem 1.4 *The map ℓ_* is a homeomorphism onto its image.*

It is thus possible to put a natural topology on $\mathcal{T} \cup \overline{\mathcal{S}}$; by the above, we consider the topological space $\ell_*(\mathcal{T}) \cup I_*(\mathcal{MF})$, in which the rays in $I_*(\mathcal{MF})$ are identified to points, taken with the quotient topology.

In chapter 7, we prove (for the case where M has no boundary) the following:

Theorem 1.5

1. The topological space $\mathcal{T} \cup \overline{\mathcal{S}}$ is homeomorphic to D^{6g-6}, if M is closed, of genus $g > 1$; it is homeomorphic to $D^{6g-g+2b}$ if M has Euler characteristic < 0 and b boundary components.

2. The canonical map $\mathcal{T} \cup \overline{\mathcal{S}} \to P(\mathbb{R}_+^{\mathcal{S}})$ is an immersion.

The space $\mathcal{T} \cup \overline{\mathcal{S}}$, notated $\overline{\mathcal{T}}$, is the **Thurston compactification** of the Teichmüller space. It follows immediately from the definitions that for any diffeomorphism ϕ of M, the natural action of ϕ on $\overline{\mathcal{T}}$ is continuous.

Thurston compactification

If ϕ is a diffeomorphism of M, and $[\phi]$ designates the homeomorphism induced by ϕ on $\mathcal{T}$, then $[\phi]$ has a fixed point, by Brouwer's theorem.

(i) If $[\phi]$ has a fixed point in $\mathcal{T}$, then ϕ is isotopic to an isometry ϕ' in a hyperbolic metric; in particular, ϕ' is periodic.

(ii) If $[\phi]$ fixes a point in $\overline{\mathcal{S}}$, there is a foliation $\mathcal{F}$ such that $\phi(\mathcal{F})$ is Whitehead equivalent to $\lambda\mathcal{F}$, $\lambda \in \mathbb{R}_+$, where $\lambda\mathcal{F}$ has the same underlying foliation as $\mathcal{F}$, with a transverse measure λ times that for $\mathcal{F}$.

This cursory analysis will be made more precise in what follows.

1.5 Pseudo-Anosov Diffeomorphisms

We begin with a very elementary example. Let $\phi \in \mathrm{Diff}_+(T^2)$. Up to isotopy, ϕ is in $\mathrm{SL}_2(\mathbb{Z})$. There are three distinct possibilities for the eigenvalues λ_1 and λ_2 of ϕ, as follows:

(a) λ_1 and λ_2 are complex, $\lambda_1 = \bar{\lambda}_2$, $\lambda_1 \neq \lambda_2$, $|\lambda_1| = |\lambda_2| = 1$. In this case, ϕ is of finite order.

(b) $\lambda_1 = \lambda_2 = 1$; respectively, $\lambda_1 = \lambda_2 = -1$. Up to a change of coordinates,

$$\phi = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \text{respectively,} \quad \phi = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix},$$

Dehn twist

that is, ϕ is a "**Dehn twist**". In either case, ϕ leaves invariant a simple curve.

Anosov

(c) λ_1 and λ_2 are distinct irrationals. Then ϕ is an **Anosov** diffeomorphism.

This analysis is generalized by Thurston to any compact surface.

Theorem 1.6 *Any diffeomorphism ϕ on M is isotopic to a diffeomorphism ϕ' satisfying one of the following three conditions:*

(i) ϕ' fixes an element of $\mathcal{T}$ and is of finite order.

(ii) ϕ' is "reducible", in the sense that it preserves a simple curve (representing an element of $\mathcal{S}'$); in this case, one pursues the analysis of ϕ' by cutting M open along this curve.

(iii) There exists $\lambda > 1$ and two transverse measured foliations $\mathcal{F}^S$ and $\mathcal{F}^U$ such that

$$\phi'(\mathcal{F}^S) = \frac{1}{\lambda}\mathcal{F}^S;$$
$$\phi'(\mathcal{F}^U) = \lambda\mathcal{F}^U;$$

These equalities indicate that the underlying foliations are equal.

Apart from the obvious case, namely when $\mathcal{F}^S$ and $\mathcal{F}^U$ are transverse, this says that their singularities are the same, and there exists a neighbourhood of each singularity which is analogous to that in Figure 1.3. A diffeomorphism

pseudo-Anosov

which satisfies condition (iii) is called **pseudo-Anosov**.

Theorem 1.6 is proved in chapter 8. In order to later use this theorem for an efficient induction, we need to extend the theory to the case with boundary. This is realized in chapter 10.6.

In chapter 11, we show that, for a pseudo-Anosov ϕ, $\mathcal{F}^S$ and $\mathcal{F}^U$ represent the only fixed points of $[\phi]$ in $\overline{\mathcal{T}}$, and the two pseudo-Anosov homotopies are conjugate by a diffeomorphism isotopic to the identity.

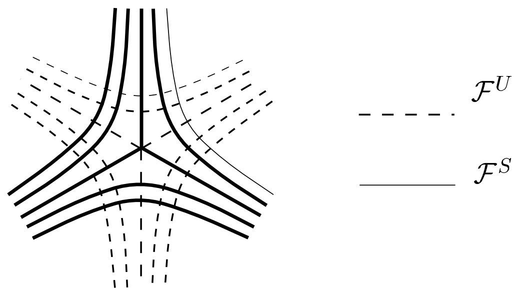


Figure 1.3: Pseudo-Anosov Singularities

The key to these theorems is the following "mixing" property which the pseudo-Anosov diffeomorphism ϕ possesses: for all $\alpha, \beta \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{i(\phi^n\alpha, \beta)}{\lambda^n} = I(\mathcal{F}^S, \alpha) I(\mathcal{F}^U, \beta)$$

1.5.1 Spectral properties of pseudo-Anosov diffeomorphisms.

For $\theta \in \mathcal{T}$ and $\alpha \in \mathcal{S}$, we defined in section 1.4 the positive number $\ell(\theta, \alpha)$. Diffeomorphisms take proper values in the following sense:

Theorem 1.7 *Let $\phi \in \mathrm{Diff}(M^2)$. There exists a finite family of algebraic integers $\lambda_1, \ldots, \lambda_k \geq 1$ such that, for every $\alpha \in \mathcal{S}$, there exists λ_j satisfying: for all $\theta \in \mathcal{T}$, $\lim_{n\to\infty} \ell(\theta, \phi^n\alpha)^{1/n} = \lambda_j$. Furthermore, ϕ is pseudo-Anosov if and only if $k = 1$ and $\lambda_1 > 1$; in this case $\lambda_1 = \lambda$. (See chapters 8 and 10.6).*

1.5.2 Entropy.

On a compact metric space X with a continuous map $f\colon X \to X$, we may define the **topological entropy**. (See chapter 9.5). If ϕ is a pseudo-Anosov diffeomorphism, one proves that $h(\phi) = \log(\lambda)$. Moreover, ϕ possesses an obvious invariant measure and $h(\phi)$ is its **metric entropy** [Sin76b].

topological entropy

metric entropy

Theorem 1.8 *A pseudo-Anosov diffeomorphism minimizes the topological entropy in its isotopy class.*

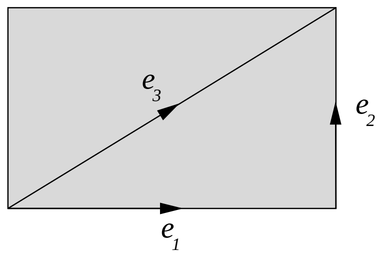


Figure 1.4: The torus T^2

The list of Thurston's results is much longer, but we end this overview here to come to the heart of the matter.

1.6 The case of the torus T^2

This case is particularly simple and is treated separately. On the torus T^2, consider the three elements e_1, e_2, e_3 in $\mathcal{S}(T^2)$, shown in figure 1.4. Let these be provisionally oriented.

Let x_1 and x_2 be the canonical generators e_1 and e_2 given the orientations shown in figure 1.4.

If γ is a simple oriented curve, $\gamma = mx_1 + nx_2$, we find

$$i(e_1, \gamma) = |n|, \quad i(e_2, \gamma) = |m|, \quad i(e_3, \gamma) = |n - m|.$$

These three numbers determine γ in $\mathcal{S}$, but the first two are not sufficient. They form a "degenerate triangle", in the sense that any one of them is equal to the sum of the other two.

We now consider the standard simplex with barycentric coordinates X_1, X_2, X_3, (where $X_i \geq 0$, $\sum X_i = 1$). This decomposes into the four regions indicated in figure 1.5.

Let $(\leq \nabla)$ be the domain where the triangle inequality holds; the boundary $\partial(\leq \nabla)$ corresponds to degenerate triangles. The standard simplex being regarded in $\mathbb{R}^3_+$, by $\mathrm{cone}(\partial(\leq \nabla))$ we mean the cone in the half-line from the center 0 supported on $\partial(\leq \nabla)$.

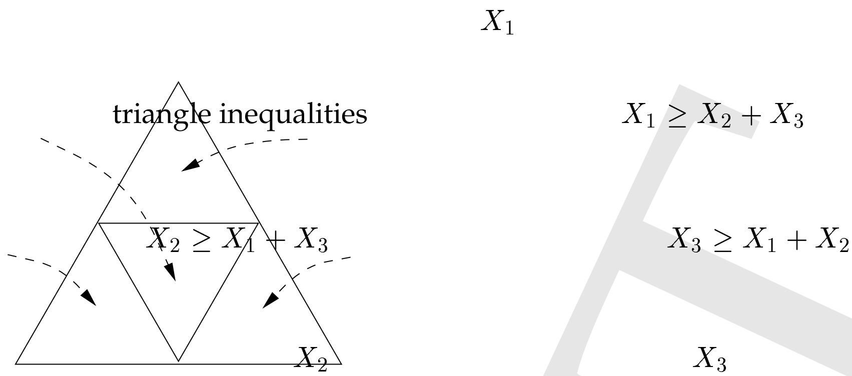


Figure 1.5:

For every $\gamma \in \mathcal{S}$, we associate the numbers

$$x_j := \frac{i(e_j, \gamma)}{\sum_j i(e_j, \gamma)}, \quad j = 1, 2, 3;$$

a simple calculation shows that we can thus identify with $\mathcal{S}$ the set of rational points of $\partial(\leq \nabla)$.

Lemma 1.9 *Let $\beta \in \mathcal{S}$. There exists a continuous function*

$$\Phi_\beta \colon \operatorname{cone}(\partial(\leq \nabla)) \to \mathbb{R}_+,$$

homogeneous of degree 1 (for multiplication by positive scalars), such that, for all $\alpha \in \mathcal{S}$,

$$i(\alpha, \beta) = \Phi_\beta\left(i(\alpha, e_1), i(\alpha, e_2), i(\alpha, e_2)\right).$$

Proof. We give an explicit construction. Suppose that β is represented by $mx_1 + nx_2, n, m \in \mathbb{Z}$, $(m, n) = 1$. (The only ambiguity in this is that $-mx_1 - nx_2$ also corresponds to β.) On the surface of the cone $X_3 = X_1 + X_2$, we have

$$\Phi_\beta(X_1, X_2, X_3) = \left|\det\begin{pmatrix} X_2 & -X_1 \\ m & n \end{pmatrix}\right|$$

On the other two faces, we have

$$\Phi_\beta(X_1, X_2, X_3) = \left|\det\begin{pmatrix} X_2 & X_1 \\ m & n \end{pmatrix}\right|$$

At the intersection of these faces, these formulas show that Φ_β has the stated property. ▮

Remark. Φ_β is piecewise linear, a property which we may recover from the other "explicit" formulas of the theory.

Consider now a sequence (λ_n, α_n) with $\lambda_n \in \mathbb{R}_+$, $\alpha_n \in \mathcal{S}$, such that, for all $\beta \in \mathcal{S}$, the sequence $\lambda_n i(\alpha_n, \beta)$ converges. Denote by $\lim(\lambda_n, \alpha_n)$ the functional

$$\lim(\lambda_n, \alpha_n)(\beta) := \lim \lambda_n i(\alpha_n, \beta).$$

Since Φ_β is homogeneous, we have

$$\lim(\lambda_n, \alpha_n)(\beta) = \Phi_\beta(\lim(\lambda_n, \alpha_n)(e_1), \lim(\lambda_n, \alpha_n)(e_2), \lim(\lambda_n, \alpha_n)(e_3)).$$

This implies that the bijection of $\mathbb{R}_+ \times \mathcal{S}$, regarded as part of $\mathbb{R}_+^{\mathcal{S}}$, onto the rational rays of $\mathrm{cone}(\partial(\leq \nabla))$ prolongs to a homogeneous homomorphism:

$$\overline{\mathbb{R}_+ \times \mathcal{S}} \simeq \mathrm{cone}(\partial(\leq \nabla)) \simeq \mathbb{R}^2$$

Thus, in $P(\mathbb{R}_+^{\mathcal{S}})$, we have $\overline{\mathcal{S}} \simeq S^1$.

Consider a measured foliation $\mathcal{F}$ on T^2. One can show that $\mathcal{F}$ has no singularities and that it is transversely orientable (this is a consequence of a simple Euler-Poincaréformula); with this measured foliation we identify a closed non-singular 1-form. This form is then isotopic to a unique *linear form* (a 1-form with constant coefficients in the canonical coordinates on T^2) [Ste69].

If ω is linear, every curve $\gamma = mx_1 + nx_2$ is transverse to ω, or else contained in a leaf; thus

$$\left| \int_\gamma \omega \right|^2 = I(\omega, \gamma);$$

ω is determined up to sign by $I(\omega, e_1)$, $I(\omega, e_2)$ $I(\omega, e_2)$. Lemma 1.10 is now clear:

Lemma 1.10 *Let $\mathcal{F}$ be a measured foliation on T^2. Then:*

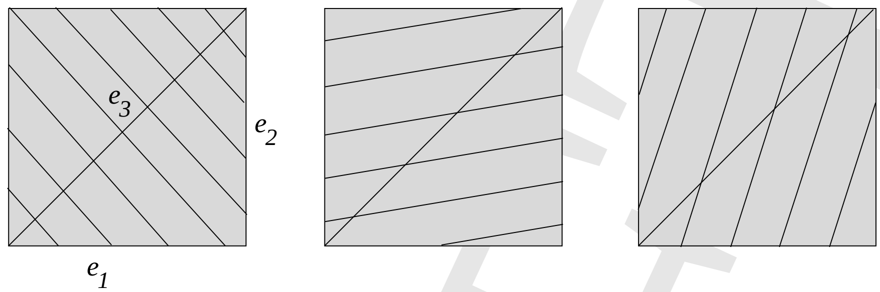
1. *$I(\mathcal{F}, e_1)$, $I(\mathcal{F}, e_2)$, $I(\mathcal{F}, e_3)$ form a degenerate triangle.*

2. *If $\beta \in \mathcal{S}$, then*

$$I(\mathcal{F}, \beta) = \Phi_\beta(I(\mathcal{F}, e_1), I(\mathcal{F}, e_2), I(\mathcal{F}, e_3)),$$

where Φ_β is the function of lemma 1.9.

The first point is clear from figure 1.6



$I(\mathcal{F}, e_3) =$ $I(\mathcal{F}, e_1) + I(\mathcal{F}, e_2)$ $\quad$ $I(\mathcal{F}, e_2) =$ $I(\mathcal{F}, e_1) + I(\mathcal{F}, e_3)$ $\quad$ $I(\mathcal{F}, e_1) =$ $I(\mathcal{F}, e_2) + I(\mathcal{F}, e_3)$

Figure 1.6: Proof of lemma 1.10–(1)

As a consequence, in $P(\mathbb{R}_+^{\mathcal{S}})$, we have $\pi I_*(\mathcal{MF}) = \overline{\mathcal{S}}$.

In section 1.4, we defined the Teichmüller space in the context of $\chi < 0$. For T^2, one may give an analogous definition, by considering the flat metric ($K = 0$) such that the area of $T^2 = 1$. (This normalization condition is useless in the hyperbolic case, where the form of an object determines its volume.)

Remark 1. Instead of this normalization, one may work with a metric which is flat up to a positive scalar.

On the other hand, if T^2 is given a complex structure, its universal covering $\widetilde{T^2}$ is isomorphic to $\mathbb{C}$ and the group of automorphisms of $\mathbb{C}$, $z \mapsto \alpha z + \beta$, $\alpha, \beta \in \mathbb{C}$, coincides with the group of orientation preserving maps of $\mathbb{R}^2$ preserving the euclidean metric up to a scalar. From this, one easily deduces the equivalence of our definition of $\mathcal{T}$ with the classical definition: "the set of complex marked structures on T^2, up to isotopy."

Remark 2. A flat structure on T^2 has an underlying affine structure. If we fix two generators e_1, e_2, for $\pi_1(T^2)$, the affine structure underlying the metric ρ

is determined in the following way: all given geodesics of the class of e_i are parallel closed geodesics; thus the numbers

$$\mathrm{dist}(\frac{\Delta}{\Delta'}) \Big/ \mathrm{dist}(\frac{\Delta'}{\Delta''}) \in \mathbb{R}_+$$

correspond to three geodesics of the system e_i. It is easy to see that all their affine structures on T^2 are isotopic to each other. Thus we may always represent an element of $\mathcal{T}$ by a flat metric ρ whose underlying affine structure is the canonical structure (this choice will always be made in the following.) In other words, the usual straight lines are the geodesics for ρ.

To $\rho \in \mathcal{T}$, we may associate (X_1, X_2, X_3), $X_j = \rho(e_j)/\sum_k \rho(e_k)$, where $\rho(e_j)$ is the length of the geodesic e_j in the metric ρ.

Lemma 1.11 *The map above is a homeomorphism* $\mathcal{T} \to \mathrm{int}(\leq \nabla)$.

Proof. It is clear that (X_1, X_2, X_3) satisfy the triangle inequality(s). Let Δ be a triangle in $\mathbb{R}^2$; every assignment of lengths to the sides satisfying the triangle inequality determines on $\mathbb{R}^2$ a flat metric compatible with the affine structure; this is invariant under the group of translations, hence induces a metric on T^2. This shows surjectivity. For injectivity, we note that two flat metrics with standard affine structures giving the same lengths to the sides of Δ are identical. The topology is left for the reader.

In other words, the composition

$$\mathcal{T} \xrightarrow{\ell_*} \mathbb{R}_+^{\mathcal{S}} \xrightarrow{\text{proj.}} \mathbb{R}_+^{(e_1,e_2,e_3)}$$

is a homeomorphism of $\mathcal{T}$ onto its image. To see that ℓ_* is a homeomorphism onto its image, note that the length of a given line segment depends continuously on the lengths assigned to e_1, e_2, e_3 (classical trigonometry!)

We have:

$$\ell_*(\mathcal{T}) \bigcap I_*(\mathcal{MF}) = \emptyset$$

Indeed, if ω is a differential form, there exists a sequence γ_n of simple closed curves such that $\int_\gamma \omega \to 0$; if α_n denotes the class of γ_n in $\mathcal{S}$, we have $I_*(\omega)(\alpha_n) \to 0$, while for a given metric the lengths of the closed geodesics do not approach zero. ∎

To prove the analogue of theorem 1.4 for the torus T^2, it remains to prove the following lemma.

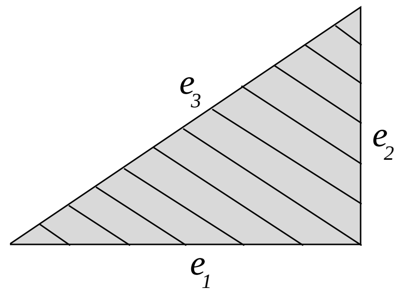


Figure 1.7:

Lemma 1.12 *Let ρ_n be a sequence of flat metrics (normalized to the canonical affine structure), λ_n a sequence of positive reals, and ω a linear form. Suppose that, for* $j = 1, 2, 3$,

$$\lambda_n \rho_n(e_j) \to \left| \int_{e_j} \omega \right|$$

Then for all closed geodesics α,

$$\lambda_n \rho_n(\alpha) \to \left| \int_{\alpha} \omega \right|$$

Proof.

Let ρ'_n denote the metric $\lambda_n \rho_n$. We treat the case where ω is on the face $X_3 = X_1 + X_2$ of $\mathrm{cone}(\partial(\leq \nabla))$ (figure 1.7) and $\int_{e_i} \omega \neq 0$ for $i = 1, 2$.

Orient $e_j, j = 1, 2, 3$, so that $\int_{e_j} \omega \geq 0$. Now let θ_n be the magnitude of the angle between e_1 and e_2 in the metric ρ'_n.

We have

$$[\rho'_n(e_3)]^2 = [\rho'_n(e_1)]^2 + [\rho'_n(e_2)]^2 + 2\rho'_n(e_1)\rho'_n(e_2)\cos\theta_n.$$

The hypothesis then implies that $\cos\theta_n$ tends to 1. If α is a linear segment, $\alpha = a_1 e_1 + a_2 e_3$, $a_1, a_2 \in \mathbb{R}$, we have

$$[\rho'_n(\alpha)]^2 = a_1^2[\rho'_n(e_1)]^2 + a_2^2[\rho'_n(e_2)]^2 + 2a_1 a_2 \rho'_n(e_1)\rho'_n(e_2)\cos\theta_n.$$

Thus,

$$[\rho'_n(\alpha)]^2 \to \left[a_1 \int_{e_1} \omega + a_2 \int_{e_2} \omega \right]^2 = \left[\int_{\alpha} \omega \right]^2$$

On T^2 the analysis of theorem 1.6 is trivial. Theorem 1.7 reduces in the case of the torus to a spectral property well known in linear algebra. ∎

Chapter 2

Some highlights of the theory of surface diffeomorphisms

2.1 The space of functionals

Let M^2 be a compact, connected manifold of dimension 2. I will consider the group of diffeomorphisms of M^2, denoted by $\mathrm{Diff}(M^2)$. If $A \subset M^2$, I will denote by $G(M, A)$ the space of homotopy equivalences $M \xrightarrow{f} M$, such that $f|A = \mathrm{id}$, with the topology of uniform convergence.

Theorem 2.1 (Smale) $\mathrm{Diff}(D^2, \mathrm{rel}\, \partial D^2)$ *is contractible (* $\mathrm{Diff}(D^2, \mathrm{rel}\, \partial D^2) \simeq *$*).*

For a proof, see [Cer68], [Sma59].

Theorem 2.2 *See [Cer68]. The following natural inclusions are homotopy equivalences:*

$$O(3) \hookrightarrow \mathrm{Diff}\, S^2 \hookrightarrow G(S^2)$$
$$SO(3) \hookrightarrow \mathrm{Diff}\, P^2 \hookrightarrow G(P^2)$$

In the usual situation, M is a $K(\pi_1, 1)$; consider $* \in M$ and the fibration

$$\begin{tikzcd} G(M, *) \arrow[r, hook] & G(M) \arrow[d, "ev(*)"] \\ & M \end{tikzcd}$$

By standard methods of obstruction theory, one proves the following theorem:

Theorem 2.3

$$\pi_i G(M,*) = \begin{cases} \mathrm{Aut}(\pi_1(M,*)) & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases}$$

Therefore, the exact homotopy sequence of our fibration reduces to

$$1 \to \pi_1 G(M) \to \pi_1 M \xrightarrow{\partial} \mathrm{Aut}(\pi_1 M) \to \pi_0 G(M) \to 1$$

One verifies without difficulty the following facts:

1) If $x \in \pi_1 M$, then $\partial(x)$ is the inner automorphism corresponding to x.

2) $\pi_1 G(M)$ is the center of $\pi_1 M$. This group is trivial except in the following exceptional cases: the torus, $\pi_1 G(T^2) = \mathbb{Z} \oplus \mathbb{Z}$; the Klein bottle: $\pi_1 G(K^2) = \mathbb{Z}$.

3) $\pi_0 G(M) = \mathrm{Aut}(M)/$the group of inner automorphisms.

2.2 The Braid groups and their computation

(See [Bir75] for more details)

big diagonal

Let X be a topological space, n a positive integer and $P_n(X) := X^n \setminus \Delta$, where Δ is the "**big diagonal**" of X^n, which is the set of n-tuples $(x_1, \ldots, x_n)$ of points of X, such that for some $i \neq j$, $x_i = x_j$. The symmetric group $\mathrm{Sym}(n)$ acts freely on $P_n(X)$ and by definition, $B_n(X) := P_n(X)/\mathrm{Sym}(n)$. One thus has a Galois (regular) covering

$$\begin{array}{ccc} \mathrm{Sym}(n) & \to & P_n(X) \\ & & \downarrow \\ & & B_n(X). \end{array}$$

By definition, $\pi_1 P_n(X) :=$ the group of colored n-braids of X, and $\pi_1 B_n(X)$ is the group of n-braids of X.

Henceforth, $X = \mathbb{R}^2$, and we write

$$\pi_1 P_n(\mathbb{R}^2) = P_n \quad \text{the colored } n\text{-braid group}$$
$$\pi_1 B_n(\mathbb{R}^2) = B_n \quad \text{the pure } n\text{-braid group}$$

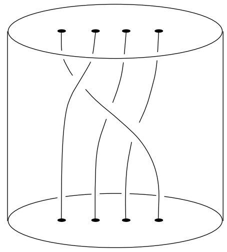


Figure 2.1: A element of B_n.

We have an obvious exact sequence:

$$1 \to P_n \to B_n \to \mathrm{Sym}(n) \to 1.$$

An element of B_n may be represented in the following manner: fix once and for all a set of n distinct points $x_1, \ldots, x_n$ in $\operatorname{int} D^2$. Then the element of B_n is a system of arcs of $D^2 \times I$, going from $(x_1, \ldots, x_n) \times 0$ to $(x_1, \ldots, x_n) \times 1$, transverse to every horizontal slice $D^2 \times t$. The arcs do not meet $\partial D^2 \times I$, and the whole is defined up to isotopy (leaving invariant the boundary of the cylinder and respecting the projection $D^2 \times I \to I$).

With this representation, the law of composition in B_n is the same as for cobordisms and the colored braids are those for which the arc leaving $x_i \times 0$ arrives at $x_i \times 1$. Figure 2.1 represents an element of B_n.

Theorem 2.4 (Fadell-Neuwirth) *The map $P_n(\mathbb{R}^2) \to P_{n-1}(\mathbb{R}^2)$ which "forgets" x_n is a fibration with fibre $\mathbb{R}^n \setminus (n-1)$points.*

Corollary 2.5

$$P_n(\mathbb{R}^2) \simeq K(P_n, 1)$$
$$B_n(\mathbb{R}^2) \simeq K(B_n, 1)$$

Remark. The theorem of Fadell-Neuwirth gives us a split short exact sequence

$$1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$$

and P_n is determined by P_{n-1} and the action of the free group F_{n-1}.

We will now give a presentation of the group B_n. In $\mathbb{R}^2$, consider the coordinates (x, y), and for $p = (p_1, \dots, p_n) \in B_n(\mathbb{R}^2)$ arrange the indices so that

$$x(p_1) \leq x(p_2) \leq \cdots \leq x(p_n).$$

By definition, $M_i \subset B_n(\mathbb{R}^2)$ is the set of p's such that $x(p_i) = x_{p-1})$, such as in figure 2.2.

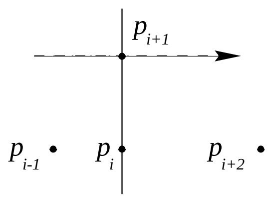


Figure 2.2: Transverse orientation of $M_i \setminus \bigcup_{j \neq i} M_j$

Note the following choices:

(1) $M_i \setminus \bigcup_{j \neq i} M_j$ is a sub-manifold of codimension 1 of $B_n(\mathbb{R}^2)$ given the canonical transverse orientation, defined as in figure 2.2. If the notation is such that $y(p_{i+1}) > y(p_i)$, a displacement of p_{i+1}, along the positive normal, pushes p_{i+1} so far that $x(p_{i+1}) > x(p_i)$.

(2) $M_i \setminus \bigcup_{j \neq i} M_j$ is connected.

(3) $B_n(\mathbb{R}^2) \setminus \bigcup_i M_i$ is contractible.

These remarks imply that the simple loops a_i, based in $B_n(\mathbb{R}^2) \setminus \bigcup_i M_i$ and such that a_i crosses M_i exactly once (and does not cross any other stratum), in the positive direction, generate B_n. One may find the relations among the a_i by considering what happens in a neighbourhood of the strata of codimension 2, where M_i and M_{ij} meet.

Case 1. $|i - j| \geq 2$. At the level of $\mathbb{R}^2$, a point of $M_i \cap M_j$ is as in figure 2.3. One may move independently along the dashed horizontal arrows, which give us a small square, transverse to $M_i \cap M_j$ in $B_n(\mathbb{R}^2)$, as shown in figure 2.4.

We give the proper orientation to the strata M_i, M_j. This gives us the relation $a_i a_j = a_j a_i$.

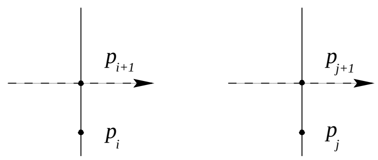


Figure 2.3:

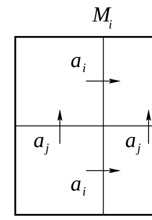


Figure 2.4:

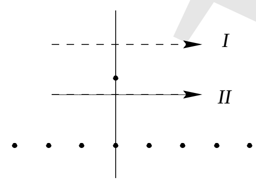


Figure 2.5:

Figure 2.6:

Case 2. $|i-j| = 1$. At the level of $\mathbb{R}^2$, we have figure 2.5, and at the level of $B_n(\mathbb{R}^2)$, figure 2.6. From these we may read off the relation:

$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}.$$

We therefore have the following theorem:

Theorem 2.6 (E. Artin) *B_n admits generators $a_1, a_2, \ldots, a_{n-1}$ and relations*

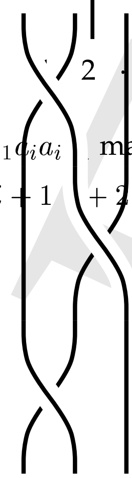
$$[a_i, a_j] = 1 \quad (|i-j| > 1)$$
$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}.$$

Corollary 2.7 $B_3 = \pi_1(S^3 \setminus \text{the trefoil knot})$.

[The explanation of this “coincidence” is this: $B_n(\mathbb{R}^2)$ may be identified with the set of complex monic polynomials of degree n, having distinct roots. Thus $B_n = \pi_1(\mathbb{C}^n \setminus$ the discriminant locus $) \ldots$].

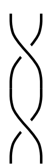
The generator a_i is the following braid:

and the relation $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$ may be visualized as follows:



In particular, $B_2 \approx \mathbb{Z}$ and the generator a_1 is

Similarly, $P_2 \approx \mathbb{Z}$ is generated by



and the natural inclusion $P_2 \hookrightarrow B_2$ is multiplication by 2: $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$.

2.3 Diffeomorphisms of the two holed disk: The spaces $A(P^2)$, $A'(P^2)$

Let $K \subset \operatorname{int} D^2$ be a finite set of cardinality k. We introduce the following notation:

$$\begin{aligned}
&\operatorname{Diff}(D^2, \operatorname{rel}(K, \partial)) := \\
&\qquad \{\text{diffeomorphisms } D^2 \xrightarrow{\varphi} D^2, \text{ such that } \varphi | K \cup \partial D^2 = \operatorname{id}\}, \\
&\operatorname{Diff}(D^2, K, \operatorname{rel} \partial) := \\
&\qquad \{\text{diffeomorphisms } D^2 \xrightarrow{\psi} D^2, \text{ such that } \psi(K) = K, \psi | D^2 = \operatorname{id}\}.
\end{aligned}$$

We have a natural action of $\operatorname{Diff}(D^2, \operatorname{rel}\partial)$ on $B_k(\operatorname{int} D^2)$, and on $P_k(\operatorname{int} D^2)$, which furnishes us with two fibrations:

$$\operatorname{Diff}(D^2, K, \operatorname{rel} \partial) \hookrightarrow \operatorname{Diff}(D^2, \operatorname{rel} \partial) \to B_k(\operatorname{int}\ D^2)$$

and

$$\operatorname{Diff}(D^2, \operatorname{rel}(K, \partial)) \hookrightarrow \operatorname{Diff}(D^2, \operatorname{rel} \partial) \to P_k(\operatorname{int}\ D^2)$$

Applying the theorem of Smale:

$$\operatorname{Diff}(D^2, \operatorname{rel} \partial) \simeq *,$$

we have the following corollary:

Corollary 2.8

1) *Every connected component of* $\operatorname{Diff}(D^2, \operatorname{rel}(K, \partial)), \operatorname{Diff}(D^2, K, \operatorname{rel}\partial)$ *is contractible.* ∎

2) *We have the canonical isomorphisms*

$$\begin{aligned}
P_k &= \pi_0(\operatorname{Diff}(D^2, \operatorname{rel}(K, \partial))) \\
B_k &= \pi_0(\operatorname{Diff}(D^2, K, \operatorname{rel} \partial))
\end{aligned}$$

We will now consider the manifold (with boundary) P^2 which is the "disk with two holes", or the "pair of pants" (see figure 2.7.)

Remark. Let

$\operatorname{Diff}(P^2, \partial_2, \partial_3, \operatorname{rel}\partial_1) = \{\varphi \in \operatorname{Diff}(P^2) : \varphi | \partial_1 P^2 = \operatorname{id}, \varphi(\partial_2 \mathrm{P}^2) = \partial_2 \mathrm{P}^2, \varphi(\partial_3 \mathrm{P}^2) = \partial_3 \mathrm{P}^2\}$. ∎

Then $\operatorname{Diff}(P^2, \partial_2, \partial_3, \operatorname{rel}\partial_1)$ is manifestly of the same homotopy type as $\operatorname{Diff}(D^2, \operatorname{rel}(K, \partial))$. ∎

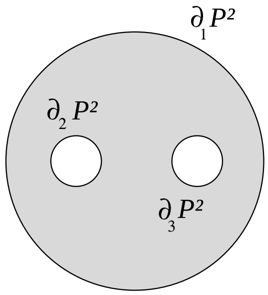


Figure 2.7: The "pair of pants" P^2

Proposition 2.9

$$\pi_0(\mathrm{Diff}(P^2, \mathrm{rel}\,\partial)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Proof. Consider the 1-jets of the diffeomorphisms, at two points of K. We obtain a fibration:

$$\begin{array}{ccc} \mathrm{Diff}(P^2, \mathrm{rel}\,\partial P^2) & \hookrightarrow & \mathrm{Diff}(D^2, \mathrm{rel}(K, \partial)) \\ & & \downarrow \\ & & S^1 \times S^1 \end{array}$$

with exact sequence

$$0 \to \pi_1(S^1 \times S^1) \to \pi_0(\mathrm{Diff}(P^2, \mathrm{rel}\,\partial P^2)) \to P_2 \to 0$$

One may verify that this sequence splits, that that extension is central, and that the action of P_2 on $\pi_1(S^1 \times S^1)$ is trivial, which gives the stated result. ∎

We now consider

$\mathrm{Diff}_+(P^2, \partial_1, \partial_2, \partial_3) :=$
$\{$orientation preserving diffeomorphisms $\varphi\colon P^2 \to P^2$ such that $\varphi(\partial_i P^2) = \partial_i P^2\}$.

Proposition 2.10 $\mathrm{Diff}_+(P^2, \partial_1, \partial_2, \partial_3)$ *is contractible.*

Proof. By restricting an element $\varphi \in \mathrm{Diff}_+(P^2, \partial_1, \partial_2, \partial_3)$ to $\partial_1 P^2 = \partial D^2$, we have a fibration:

$$\begin{array}{ccc} \underbrace{\mathrm{Diff}(P^2, \partial_2, \partial_3, \mathrm{rel}\ \partial_1)}_{P^2 = K(\mathbb{Z},0)} & \hookrightarrow & \mathrm{Diff}_+(P^2, \partial_1, \partial_2, \partial_3) \\ & & \Big\downarrow \text{restriction} \\ & & \mathrm{Diff}_+ S^1 = K(\mathbb{Z}, 1) \end{array}$$

One verifies also that the arrow

$$\pi_1 \mathrm{Diff}_+ S^1 \xrightarrow{\partial} \pi_0(P^2, \partial_2, \partial_3, \mathrm{rel}\ \partial_1) = P_2$$

is an isomorphism, which gives the result. ∎

Now let N^2 be any compact surface, with non-empty boundary. Define $A(N^2) :=$ isotopy classes of arcs $I \subset N^2$, with $\partial I \subset \partial N^2$, each end free to move on the respective connected component of ∂N^2, and representing the non-trivial elements of $\pi_1(N^2, \partial N^2)$, and let

$$A'(N^2) := \{\text{the same as above but with several pairwise disjoint arcs}\}.$$

Corollary 2.11 *$A(P^2)$ consists of exactly six elements, classified by the connected components of ∂P^2, in which the ends of the respective arcs fall.*

Proof. Let τ and τ' be two representatives of elements of $A(P^2)$ with ends in the same connected component of ∂P^2. We may easily check that there is an orientation preserving diffeomorphism

$$(P^2, \tau) \xrightarrow{\psi} (P^2, \tau').$$

Since $\pi_0 \mathrm{Diff}_+(P^2, \partial_1, \partial_2, \partial_3) = 0$, this diffeomorphism is isotopic to the identity, which gives the result. The six models are given in figure 2.8. ∎

Now let A' be the set of ordered triples (a_1, a_2, a_3), where $a_i \geq 0$, $a_i \in \mathbb{Z}_+$, and $\sum_i a_i \equiv 0$ (mod 2). To $\tau \in A'(P^2)$, associate

$$i(\tau) = (i(\tau, \partial_1), i(\tau, \partial_2), i(\tau, \partial_3)) \in A'$$

where $i(\tau, \gamma)$ is the number of points τ has in common with γ. For convenience, we adjoin $\emptyset \in A'(P^2)$, with $i(\emptyset) = (0, 0, 0)$.

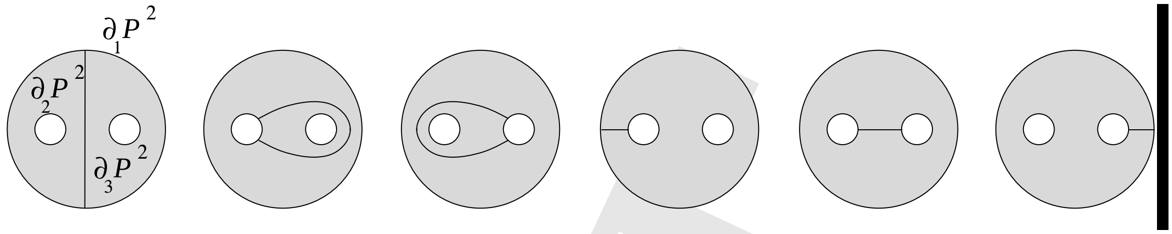


Figure 2.8: The six models for $A(P^2)$.

Theorem 2.12 *The map $A'(P^2) \xrightarrow{i} A'$ is a bijection.*

Proof. We begin by constructing a map $A' \xrightarrow{\tau} A'(P^2)$ such that $i(\tau(a_1, a_2, a_3)) = (a_1, a_2, a_3)$. If $(a_1, a_2, a_3) \neq 0$, then the point with barycentric coordinates

$$(\frac{a_1}{\sum a_i}, \frac{a_2}{\sum a_i}, \frac{a_3}{\sum a_i})$$

falls in one of the four regions of figure 2.9.

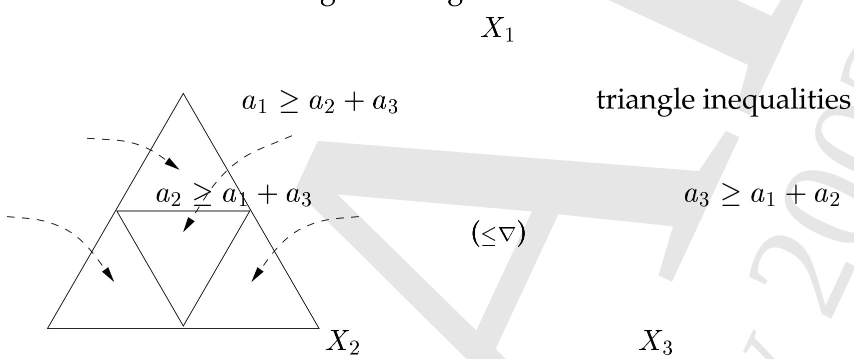


Figure 2.9:

If (a_1, a_2, a_3) satisfies the triangle inequalities, we consider the non-negative integers

$$x_{12} := \frac{1}{2}(a_1 + a_2 - a_3), \quad x_{23} := \frac{1}{2}(a_2 + a_3 - a_1), \quad x_{31} := \frac{1}{2}(a_3 + a_1 - a_2)$$

and we define $\tau(a_1, a_2, a_3)$ to be the element of $A'(P^2)$ which consists of $x_{ij} = x_{ji}$ segments of the type τ_{ij}, for $i \neq j$.

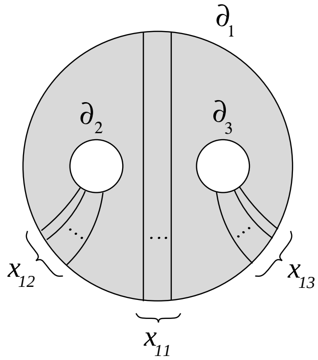


Figure 2.10:

If $a_1 \geq a_2 + a_3$, we set

$$x_{11} := \frac{1}{2}(a_1 - a_2 - a_3), \quad x_{12} := a_2, \quad x_{13} := a_3,$$

and we define $\tau(a_1, a_2, a_3)$ as in figure 2.10.

The other cases are treated in a similar manner. One may verify that on $\partial(\leq \nabla)$ the different definitions agree and that $i \circ j$ is the identity. Thus i is *surjective*.

We now observe that the compatible pairs of elements of $A'(P^2)$ are exactly those which are joined by a segment in figure 2.11.

The four triangles in figure 2.11 correspond canonically with the four triangles of figure 2.9. More precisely, let $x_{\alpha,\beta}$ be the number of segments of type $\tau_{\alpha,\beta}$ which appear in $\tau \in A'(P^2)$. We have the following four mutually exclusive situations:

1) $x_{\alpha,\alpha} = 0$ for $\alpha = 1, 2, 3$, which implies that $i(\tau) \in (\leq \nabla)$.

2) $x_{11} \neq 0$, which implies that $a_1 > a_2 + a_3$.

....

Suppose now that $\tau_1, \tau_2 \in A'(P^2)$ and that $i(\tau_1) = i(\tau_2)$. We have previously deduced that τ_1 and τ_2 are in the same one of the four situations described above; by a calculation of linear algebra .... which are (by definition) the same for τ_1 and τ_2, we conclude that the $x_{\alpha,\beta}$ are also the same. We

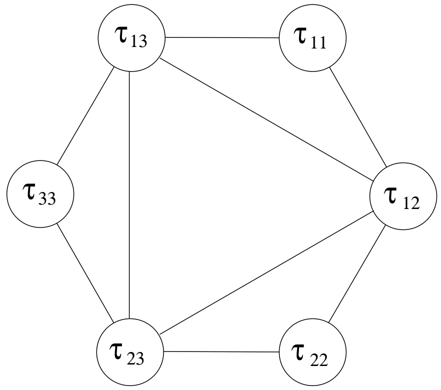


Figure 2.11:

still have to prove that if $\tau_1, \tau_2 \in A'(P^2)$ are such that for these $x_{\alpha,\beta}$ are equal, then $\tau_1 = \tau_2$. The proof of the general case is an induction on $\sum_{\alpha\leq\beta} x_{\alpha,\beta}$. We leave the details to the reader. We have thus proved that i is *injective*. ∎

Remark. Let $\tau \in A'(P^2)$. There does not exist any non-trivial element of $\pi_0\mathrm{Diff}_+(P^2, \partial_1, \partial_2, \partial_3, \tau)$. In particular, for a given τ, one may not interchange the different connected components of τ among themselves.

Chapter 3

Highlights of Hyperbolic Geometry in Dimension 2 and Generalities on $i\colon \mathcal{S}\times\mathcal{S}\to\mathbb{R}_+$

by V. Poénaru

3.1 A Little Hyperbolic Geometry

Consider a compact surface M, with a riemannian metric of curvature -1, for which the boundary is geodesic if it is non-empty. The universal covering $\widetilde{M}$ is isometric to a domain in the hyperbolic plane $\mathbb{H}^2$, possibly bounded by geodesics of $\mathbb{H}^2$.

Lemma 3.1 *Let α and β be distinct geodesic arcs in M, with the same endpoints. Then the closed curve $\alpha\cup\beta$ is not homotopic to zero.*

Proof. If $\alpha\cup\beta$ is homotopic to zero, then it lifts to a closed curve in $\widetilde{M}$. But two geodesics in $\mathbb{H}^2$ may not meet in two points. This property of $\mathbb{H}^2$ results, for example, in the Gauss-Bonnet formula: for a disk D with a riemannian metric so that the boundary is a geodesic polygon, we have

$$\int\!\!\int_C K = 2\pi - \sum \text{exterior angles}$$

where K denotes the curvature. ∎

Lemma 3.2 *Let V be a compact riemannian manifold with a totally geodesic boundary. Then in every (free) homotopy class of maps $S^1\to V$ there is a geodesic immersion, having the minimum length of loops within its homotopy class.*

Proof.

We take a homotopy class $\alpha \in [S^1, V]$, and set $L := N\epsilon$ where $\epsilon > 0$, and the integer N are chosen as follows: ϵ should be smaller than the injectivity radius of the exponential map, and N large enough so that α contains at least a curve of length $\leq L$.

Let $I(\alpha, \epsilon, N)$ be the space of continuous maps $S^1 \to M$ in the class α, composed of at least N geodesic arcs of length $\leq \epsilon$ each. This space, with the compact-open topology, is compact. The length function y is continuous. Let ϕ be a curve which realizes the minimum length in $I(\alpha, \epsilon, N)$. It is easy to check that ϕ is in fact smooth (if $\partial V \neq \emptyset$, the hypothesis that ∂V is totally geodesic intervenes here.)

To see that the length of ϕ is a minimum for the class α, it suffices to remark that if C is a rectifiable curve in α, of length $\leq L$, there exists a curve belonging to $I(\alpha, \epsilon, N)$ of length less than or equal to that of C. ∎

Remark. Without compactness, with only the hypothesis that the metric is complete, one finds that all elements of $\pi_1(V, x_0)$ are realizable by closed geodesics which, in general, are not smooth in x_0.

Lemma 3.3 *For every covering transformation T of $\widetilde{M}$ over M, there exists a unique geodesic invariant under T. It is the lift of the closed smooth geodesic in M which is in the free homotopy class α of $\pi_1(M, x_0)$ corresponding to T.*

Proof.

Existence. Here is a proof which does not make use of the hypothesis on curvature. We take as a model for $\widetilde{M}$ the set of continuous paths $\{\phi\colon [0,1] \to M \mid \phi(0) = x_0\}$ quotiented by the relation $\phi \sim \psi$ if ϕ is homotopic to ψ with endpoints fixed. The projection $p\colon \widetilde{M} \to M$ is given by $\phi \mapsto \phi(1)$. The constant path defines the basepoint in $\widetilde{M}$. Let $\psi \in \widetilde{M}$, $p(\psi) = y$ and let χ be a path in M such that $\chi(0) = y$; the lifts of χ in $\widetilde{M}$ starting from ψ is a one parameter family of paths in M, obtained by truncating the path $\psi * \chi$; this family begins with ψ and ends with $\psi * \chi$. The left action of $\pi_1(M, x_0)$ on $\widetilde{M}$ is defined as follows. For $\alpha \in \pi_1(M, x_0)$, represented by a loop ϕ, and for $\psi \in \widetilde{M}$, we set $T_\alpha(\psi) := \phi * \psi$.

This being done, consider the element α for which $T = T_\alpha$ By lemma 3.2, the free homotopy class of α contains a smooth closed geodesic g_1. Let x_1 be a point in the image of g_1 and λ a path joining x_0 to x_1; this is chosen so that $\lambda * g_1 * \lambda^{-1}$ belongs to α. If $\widetilde{\lambda * g_1}$ is the lift of $\lambda * g_1$ starting from the base point of $\widetilde{M}$, we have

$$\widetilde{\lambda * g_1}(1) = \lambda * \widetilde{g_1 * \lambda}^{-1} = T_\alpha(\tilde{\lambda}(1)).$$

Then, if we take in $\widetilde{M}$ the image of $\widetilde{\lambda * g_1}$ and all of its translates by T_{α^n}, $n \in \mathbb{Z}$, we construct a connected component of $p^{-1}(\lambda * g_1)$, consisting of a geodesic $g \in \widetilde{M}$ and lifted segments of λ, as in figure 3.1. By construction, g is invariant under T_α.

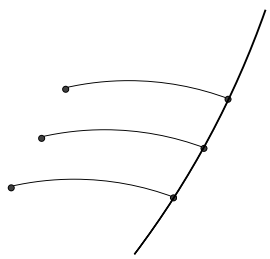


Figure 3.1:

A second proof of existence which utilizes the fact that M is a compact surface with a hyperbolic structure is the following: The transformation T_α is an isometry of $\mathbb{H}^2$. Since T_α does not have any fixed points, it is not elliptic. On the other hand, if ϕ is a parabolic isometry of $\mathbb{H}^2$ (having a unique fixed point on the circle at infinity), then for all $\epsilon > 0$ there is an $x \in \mathbb{H}^2$ such that $d(x, \phi(x)) < \epsilon$. If T_α is parabolic, this implies the existence of closed geodesics of arbitrarily short length on M, which is forbidden by compactness of M. Hence T_α is hyperbolic (having two fixed points on the circle at infinity); the geodesic g in $\mathbb{H}^2$ joining these fixed points is hence invariant under T_α. Hence g/T_α is a smooth closed geodesic in the same free homotopy class as α.

Unicity. Let g_1 and g_2 be two distinct geodesics in $\widetilde{M}$, invariant under T. If $g_1 \cap g_2$ is non-empty, the intersection consists of a single point, which is invariant under T; but this is impossible.

Hence $g_1 \cap g_2 = \emptyset$. Let $x \in g_1$; at x, we drop a perpendicular to g_2; we denote by δ the resulting geodesic segment. We note that $T\delta \cap \delta = \emptyset$, since otherwise we have a geodesic triangle the sum of whose interior angles is $> \pi$.

Now $g_1, g_2, \delta, \mathcal{T}\delta$ form a quadrilateral for which the interior angles sum to 2π (see figure 3.2), but this is impossible by the Gauss-Bonnet formula. (This can also be seen from more elementary reasons). ∎

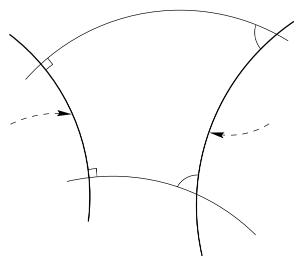


Figure 3.2:

Lemma 3.4 *Let α be a non-trivial element of $\pi_1(M, x_0)$. Then there exists a* unique *smooth closed geodesic in the homotopy class of α.*

Proof. Existence is already assured by lemma 3.2. Suppose that g_1 and g_1' are two such geodesics. The "existence" part of the preceding proof gives us two distinct T_α-invariant geodesics in $\widetilde{M}$.

But the uniqueness argument of the preceding lemma tells us that precisely that this is impossible: just apply the fact that $\pi_1(M, x_0)$ is torsion free. ∎

3.2 The Teichmüller space of the two-holed disk

The pair-of-pants P^2 (the two-holed disk) is the fundamental building block in the theory of surfaces. We showed (in exposé 2) that $\mathrm{Diff}_+(\mathrm{P}^2, \partial_1, \partial_2, \partial_3)$ is contractible; in particular, a diffeomorphism which preserves orientation and maps each boundary component to itself is isotopic to the identity.

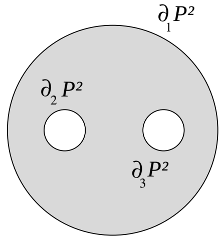


Figure 3.3:

If ρ is a metric of curvature -1 on P^2, for which every boundary component is geodesic, we say that (P^2, ρ) is a P^2-Teichmüller surface. By definition, two surfaces (P^2, ρ) and (P^2, ρ') are equivalent if there is a diffeomorphism ϕ of P^2, isotopic to the identity, such that $\phi^*\rho = \rho'$. We see that $\mathrm{Diff}_+(\mathrm{P}^2, \partial_1, \partial_2, \partial_3)$ is connected, the set of equivalence classes, which we define to be the **Teichmüller space** $\mathcal{T}(P^2)$ of P^2, may be identified with the quotient $\mathcal{H}(P^2)/\mathrm{Diff}_+$, where $\mathcal{H}(P^2)$ is the space of riemannian metrics of curvature -1 for which the boundary is geodesic: Teichmüller space $\mathcal{T}(P^2)$

$$\mathcal{T}(P^2) = \mathcal{H}(P^2)/\mathrm{Diff}_+.$$

We give $\mathcal{H}(P^2)$ the C^∞ topology and $\mathcal{T}(P^2)$ the quotient topology. We have a continuous natural map

$$L\colon \mathcal{H}(P^2) \to (\mathbb{R}_+^*)^3 = \{\text{ triples of numbers} > 0\}$$

defined by

$$L(\rho) = (\ell_\rho(\partial_1 P^2), \ell_\rho(\partial_2 P^2), \ell_\rho(\partial_3 P^2)),$$

where ℓ_ρ designates the length in the metric ρ. This induces a map which we denote by the same letter

$$L\colon \mathcal{T}(P^2) \to (\mathbb{R}_+^*)^3.$$

Theorem 3.5 *The map* $L\colon \mathcal{T}(P^2) \to (\mathbb{R}_+^*)^3$ *is a* homeomorphism. *Moreover,* $L\colon \mathcal{H}(P^2) \to (\mathbb{R}_+^*)^3$ *admits continuous local sections.*

The classification of the P^2-Teichmüller surfaces reduces to the classification of right hyperbolic hexagons, for a hyperbolic pair-of-pants may be constructed naturally by isometrically gluing two such hexagons as indicated in lemma 3.7. For the other direction, an "abstract" hyperbolic hexagon X, whose every angle is right and whose boundary components are all geodesic is isomorphic to a hexagon in the hyperbolic plane $\mathbb{H}^2$; to see this, we use X as a fundamental domain, and use symmetries on the sides of X to construct a complete, simply connected hyperbolic manifold Y; by a classical theorem (Hadamard-Cartan, [CE75]), Y is isometric to $\mathbb{H}^2$. Therefore, we focus our attention on the set Hex of (direct) isometry classes of right hexagons in $\mathbb{H}^2$ which have geodesic boundary, and which are equipped with a distinguished vertex. We write $a_1, b_1, a_2, b_2, a_3, b_3$ for the sides, named in clockwise sequence starting from the base vertex (see figure 3.4).

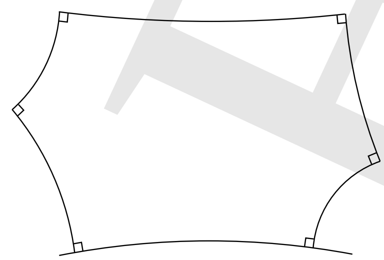


Figure 3.4:

Lemma 3.6 *The lengths* $\ell(a_1), \ell(a_2), \ell(a_3)$ *establish a bijection from* Hex *to* $(\mathbb{R}_+^*)^3$.

Proof.

Existence. Let $\ell_1, \ell_2, \ell_3 > 0$. We will construct a hexagon X in $\mathbb{H}^2$ such that $\ell(a_i) = \ell_i$ for $i = 1, 2, 3$.

We start by fixing three geodesics G, G', G'' as in figure 3.5; G and G'' are a distance ℓ_1 apart. Let $x \in G$ and let L_x be the perpendicular to G at x; if x is sufficiently far from x_0, then L_x never meets G'' again (We suggest that the reader sketch the picture in the Poincaré model). Let $x(\ell_1)$ be the point of G closest to x_0 satisfying

$$L_{x(\ell_1)} \cap G'' = \emptyset.$$

We put $f(\ell_1) := d(x_0, x(\ell_1))$.

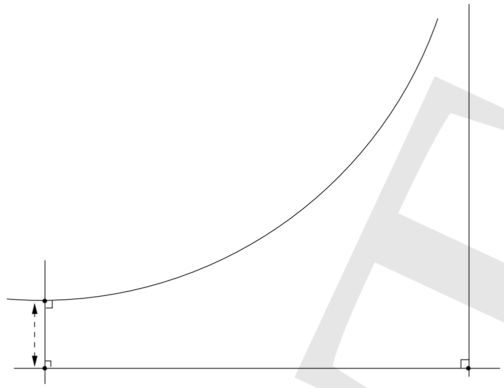


Figure 3.5:

We have sketched the construction in figure 3.6; it is determined up to isotopy by the numbers ℓ_1, ℓ_3 and λ.

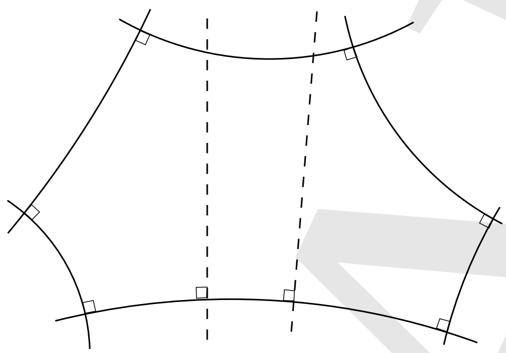


Figure 3.6:

Let $\mu(\lambda)$ be the distance from G_1'' to G_2''; this is a continuous function of the length λ, such that $\mu(0) = 0$ and $\mu(+\infty) = +\infty$ (to vary λ, we utilize the fact that there exists a one-parameter group of isometries of $\mathbb{H}^2$, leaving G invariant); μ takes every positive value. This proves the existence of X.

Unicity. As we have seen in the preceding, giving three consecutive sides of a hexagon determines it completely.

Thus if the right hexagons X and X' in figure 3.7 satisfy $\ell_i = \ell(a_i) = \ell(a_i')$

and are *not* isometric, then the lengths $\ell(b_3)$ and $\ell(b_3')$ are not equal; say that $\ell(b_3') > \ell(b_3)$.

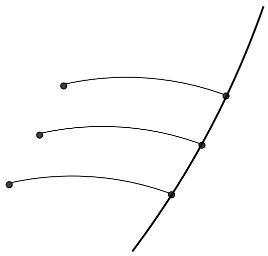


Figure 3.7:

It is simple exercise in hyperbolic geometry to see that there exists a (unique) perpendicular of b_3 to a_2 in X. This decomposes the lengths of b_3 and a_2 as shown in figure 3.8: $\ell(b_3) = \alpha + \beta$; $\ell(a_2) = \gamma + \delta$.

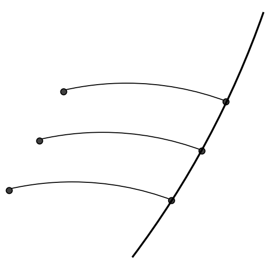


Figure 3.8:

In X', we erect perpendiculars to b_3' at the distances α and β from the two endpoints, as shown in figure 3.9. In this figure, all of the marked angles of one stretch are equal to $\pi/2$; the others are not necessarily.

Figure 3.9 gives a contradiction, since we have $\gamma + \delta > \gamma + \delta$. ∎

Remarks.

(1) The unicity which we have shown may be interpreted as the following fact: $\ell(a_1)$ and $\ell(a_2)$ being fixed, the function $\ell(b_3) \to \ell(a_2)$ is monotone;

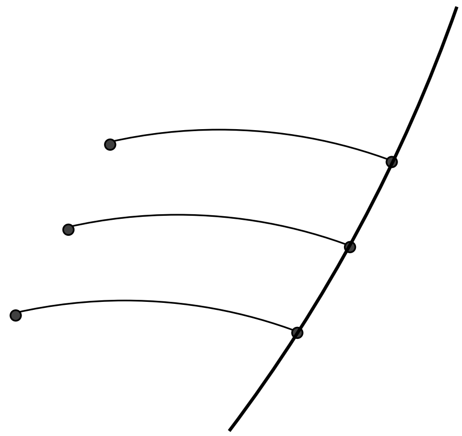


Figure 3.9:

furthermore, the function $\lambda \to \mu(\lambda)$ (figure 3.6) is a homeomorphism of $\mathbb{R}_+$.

(2) In the notation of figure 3.4, we may parameterize the set Hex by $(\ell(a_1), \ell(a_2), \ell(a_3))$ or by $(\ell(b_1), \ell(b_2), \ell(b_3))$. The transition from one set of coordinates to the other is by means of a homeomorphism of $(\mathbb{R}_+^*)^3$. [In effect, we will view this transition from $(\ell(a_1), \ell(a_2), \ell(a_3))$ to $(\ell(b_1), \ell(b_2), \ell(b_3))$ as being realized by a homeomorphism of $(\mathbb{R}_+^*)^3$. In the following, we may easily verify that the same thing is true for the transition from $(\ell(a_1), \ell(b_2), \ell(a_3))$ to $(\ell(b_3), \ell(a_2), \ell(a_3))$, etc ...]

(3) In figure 3.6, we see that if $\ell_1 := \ell(a-1)$ and $\ell_3 := \ell(a_3)$ are fixed, and if $\mu := \ell(a_2)$ tends to 0, then $\ell(b_1)$ and $\ell(b_2)$ tend to ∞.

The classification of right hexagons leads to a classification of pairs-of-pants, since every P^2-Teichmüller surface is the double of a hexagon, as indicated more precisely in the statement of lemma 3.7.

Lemma 3.7 *Given a P^2-Teichmüller surface:*

(1) there exists a unique simple geodesic g_{ij} of P^2 which joins $\partial_i P^2$ to $\partial_j P^2$ and which is perpendicular to the two. The arcs g_{12}, g_{13}, and g_{23} are mutually disjoint (figure 3.10).

(2) On $\partial_1 P^2$, the endpoints of g_{12} and g_{13} cut the segments into equal lengths. The same is true for $\partial_2 P^2$ and $\partial_3 P^2$.

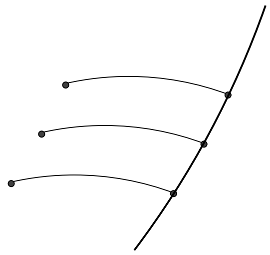


Figure 3.10:

Proof. A path of shortest length joining $\partial_i P^2$ to $\partial_j P^2$ meets the boundary at right angles at its endpoints (the first variation formula [CE75]). We deduce immediately that such a path is simple. For unicity, we remark that the homotopy class is fixed by the condition that the path be simple; by an argument using negative curvature as in lemma 3.3, we obtain the result (1).

(2) The arcs g_{12}, g_{13} and g_{23} cut P^2 into two right hexagons. These are isometric since they have three sides equal. ∎

Proof of theorem 3.5

(1) Existence. Given lengths $\ell_1, \ell_2, \ell_3 > 0$, we may construct a unique right hexagon X with $\ell(a_i) = \ell_i/2$ for $i = 1, 2, 3$ (lemma 3.6.) To for the pair-of-pants, we take two copies of X, which we glue together along b_1, b_2, and b_3. Thus, we have $\ell(\partial_i P^2) = 2\ell(a_i) = \ell_i$. This shows the surjectivity of L.

(2) Unicity. Let $\rho', \rho'' \in \mathcal{H}(P^2)$, such that $\ell_i = \ell_{\rho'}(\partial_i P^2) = \ell_{\rho''}(\partial_i P^2)$, for $i = 1, 2, 3$. We will prove that there exists $f \in \mathrm{Diff}_+(\mathrm{P}^2, \partial_1, \partial_2, \partial_3)$ which transports ρ' to ρ''.

By lemma 3.7 $(P^2, \rho') = X_1' \cup X_2'$, and $(P^2, \rho'') = X_1'' \cup X_2''$, where X_1', X_2', X_1'', X_2'' are right hexagons, parameterized by $(\ell_1/2, \ell_2/2, \ell_3/2)$. Hence, there exists a direct isometry of X_1' to X_1'' and X_2' to X_2''; the sought-for f is the "union" of these two isometries.

(3) Continuity. We will see that the continuous map

$$L\colon \mathcal{T}(P^2) \to (R_+^*)^3$$

is bijective. To show that L^{-1} is continuous, it suffices to show that $L\colon \mathcal{H}(P^2) \to (\mathbb{R}_+^*)^3$ admits continuous local sections. The easiest way is to change coordinates in $(\mathbb{R}_+^*)^3$, passing from the lengths of the boundary curves to the lengths $\ell_{12}, \ell_{23}, \ell_{13}$ of the geodesics g_{12}, g_{23}, g_{13} (figure 3.10). This gives a new continuous map

$$\Lambda\colon \mathcal{H}(P^2) \to (R_+^*)^3;$$

and it is sufficient to prove that Λ has continuous local sections.

We begin with a few preliminaries. Let E be the portion of $\mathbb{R}^2$ which is the union

$$E_0 := \{-1 \le y \le 1, x = 0\} \cup E_1 := \{-1 \le y \le 1, 0 \le x \le 1\}.$$

We define $C^\infty(E)$ to be the set of functions $f\colon E \to \mathbb{R}$ such that $f|E_0 \in C^\infty(E_0)$ and $f|E_1 \in C^\infty(E_1)$. We have a natural topology on $C^\infty(E)$ coming from the C^∞ topologies $C^\infty(E_0)$ and $C^\infty(E_1)$.

Lemma 3.8 *There is a continuous map $\epsilon\colon C^\infty(E) \to C^\infty(\mathbb{R}^2)$ such that*

$$\epsilon(f)|E = f.$$

Proof. Let $f \in C^\infty(E)$. By applying a result of Seeley [See64], we may extend the normal derivative of $f|E_0 \cap E_1$ to all of E_0. This gives us a first extension of $C^\infty(E)$ in the infinite Whitney jets on E (we use here the fact that E_0 and E_1 are in regular position). We now apply the Whitney extension theorem [Mal66]. ∎

By definition, a **truncated hexagon** is a set composed from the boundary of a C^∞ hexagon of $\mathbb{R}^2$ and of the collared neighbourhoods of three alternating sides (figure 3.11). truncated hexagon

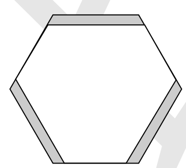


Figure 3.11: Z = truncated hexagon.

The C^∞ structure of the truncated hexagon Z is locally like that of E (wherever there are no problems). Lemma 3.8 together with some classical geometry gives the following lemma.

Lemma 3.9 *Let* $\mathrm{Emb}(Z, \mathbb{R}^2)$ *be the set of* C^∞ *embeddings of* Z *in* $\mathbb{R}^2$*, with the* C^∞ *topology. If* $\phi\colon (\mathbb{R}^n, 0) \to \mathrm{Emb}(Z, \mathbb{R}^2)$ *is a* C^∞ *function germ, we may lift* ϕ *to a germ* $\Phi\colon (\mathbb{R}^n, 0) \to \mathrm{Diff}(\mathbb{R}^2)$ *such that* $\Phi(0) = \mathrm{Id}$ *and* $\phi(t) = \Phi(t)\phi(0)$*.*

Now let $l^0 := (l^0_{12}, l^0_{23}, l^0_{13}) \in (\mathbb{R}^*_+)^3$ and let $X(l^0)$ be a right hyperbolic hexagon in $\mathbb{H}^2$ parameterized by l^0. Let G_1 and G_2 be two geodesics carrying two consecutive sides of $X(l^0)$. For l near l^0 in $(\mathbb{R}^*_+)^3$, we consider the hexagon $X(l)$ lying on $G_1 \cup G_2$ with $X(l^0)$ (figure 3.12). For every l, the double of $X(l)$ along the "marked" sides (those whose lengths are parameterized by ℓ_{ij} is a hyperbolic manifold, denoted by $2X(l)$; it is diffeomorphic to P^2.

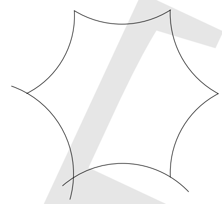


Figure 3.12:

The question is to find a diffeomorphism $\overline{\psi}(l)\colon 2X(l) \to 2X(l^0)$, so that the metric $\rho(l)$, the natural image of $2X(l)$ under $\overline{\psi(l)}$, depends continuously on l as elements of $\mathcal{H}(2X(l^0))$.

For small fixed $\epsilon > 0$ (independent of l), we consider in $X(l)$ the geodesic collars of radius ϵ along the marked sides; we thus associate to $X(l)$ a truncated hexagon $Z(l)$. Every rectangle of $Z(l)$ is foliated on the one hand by the geodesics orthogonal to the sides of the hexagon, and on the other by the orthogonal trajectories of these geodesics. It is easy to construct a continuous function germ

$$\phi\colon ((\mathbb{R}^*_+)^3), l^0) \to \mathrm{Emb}(Z(l^0), \mathbb{R}^2)$$

such that

1. $\phi(l^0)$ is the standard embedding;
2. $\phi(l)[Z(l^0)] = Z(l)$;
3. $\phi(l)$ respects the names of the marked sides and the foliations of the rectangles.

By lemma 3.9, there exists a germ

$$\psi\colon ((\mathbb{R}_+^*)^3), l^0) \to \mathrm{Emb}(X(l^0), \mathbb{R}^2)$$

such that $\psi(l)|Z(l^0) = \phi(l)$. Condition (3) assures that $2\psi(l)$ is a diffeomorphism of the doubles $2X(l^0) \to 2X(l)$. On the other hand, the construction assures that the metric on $X(l^0)$, obtained from the natural metric on $X(l)$ via $\psi(l)$, depends continuously on l. Therefore $\psi(l) := [2\psi(l)]^{-1}$ has all of the required properties. ∎

3.3 Generalities on Geometric Intersection of Simple Curves and on $i\colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$

In what follows, M is an orientable surface of genus $g > 1$. For practicality, we do not explain any case except where M is closed; the adaptations to the case of a non-empty boundary are left to the reader. I consider the set $\mathcal{S}$ of isotopy classes of simple curves on M not homotopic to zero. For $\alpha, \beta \in \mathcal{S}$, we define $i(\alpha, \beta)$ as the minimal number of points of intersection of a representative for α with a representative for β. We are led to a map

$$i_*\colon \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}}$$

Throughout this exposé, we shall make use many times of the following theorem due to D. Epstein [Eps66b]. *Let $f_0\colon S^1 \to M$ be an two-sided embedding (i.e. with trivial normal fibre) which does not bound a disk; if g_1 is an embedding homotopic to f_0, then f_0 and f_1 are isotopic. [With a base point, the same thing is true if, additionally, f_0 is not the boundary of a Möbius band.]*

In the same article, one finds the relative version: *If N is a surface with boundary and if A, B are two embedded arcs with $\partial A = \partial B = A \cap \partial N = B \cap \partial N$, homotopic with endpoints fixed, the A and B are isotopic rel ∂.*

We also use the following two facts which may be found in [Eps66b].

Every simple curve homotopic to zero in a surface with boundary bounds a disk. (This is a consequence of the Jordan-Schönflies theorem.)

On a surface, a two-sided embedding of the circle is not homotopic to any map which covers k-fold, for $k > 1$, a simple curve with two sides.

Proposition 3.10 *Let α'_0 and α'_1 be two transverse simple curves in M, not homotopic to zero. We suppose that their isotopy classes α_0 and α_1 are distinct. Then the following conditions are equivalent.*

(1) $\operatorname{card}(\alpha'_0 \cap \alpha'_1) = i(\alpha_0, \alpha_1)$.

(2) All simple closed curves formed from an arc of α'_0 and an arc of α'_1 are not homotopic to zero in M.

(3) If $\widetilde{\alpha_0}$ and $\widetilde{\alpha_1}$ are the connected components of $p^{-1}(\alpha'_0)$, respectively $p^{-1}(\alpha'_1)$, in the universal covering $p\colon \widetilde{M} \to M$ then $\operatorname{card}(\widetilde{\alpha_0} \cap \widetilde{\alpha_1}) \leq 1$.

(4) There exists in M a riemannian metric ρ of curvature -1, such that α'_0 and α'_1 are geodesics.

Proof. The reader will notice that the following implications are immediate.

$(1) \Longrightarrow (2)$; in effect, a simple closed curve γ of $\alpha'_0 \cup \alpha'_1$ which is homotopic to zero in M, is the boundary of a disk D; furthermore, γ is the union of an arc of α_0 and of an arc α'_1; to cross the disk, we may make an isotopy of α'_1 which diminishes its number of points of intersection with α'_0.

$(3) \Longrightarrow (2)$ by the theory of covering spaces.

$(4) \Longrightarrow (2)$ and (3) by lemma 3.1. ∎

Lemma 3.11 *If $\operatorname{card}(\alpha'_0 \cap \alpha'_1) > i(\alpha_0, \alpha_1)$, there exist two distinct points q_1 and q_2 of $\alpha'_0 \cap \alpha'_1$ and two (not necessarily simple) paths Γ_0, Γ_1 joining q_0 to q_1, respectively on α'_0 and α'_1, such that the singular loop $\Gamma_0 * \Gamma_1^{-1}$ is homotopic to zero in M. Hence $(3) \Longrightarrow (1)$.*

Proof. By hypothesis, there is a homotopy $h_t\colon S^1 \to M$, for $t \in [0, 1]$, such that h_0 parameterizes α'_0 and such that $h_1(S^1)$ satisfies

$$\operatorname{card}(h_1(S^1) \cap \alpha'_1) < \operatorname{card}(\alpha'_0 \cap \alpha'_1).$$

We may suppose that the isotopy h_t is in general position with respect to α'_1; that is to say that $h\colon S^1 \times [0, 1] \to M$ is transverse to α'_1. Then $h^{-1}(\alpha'_1$ is a submanifold of dimension 1 transverse to the boundary, which possesses four types of connected components, represented in figure 3.13.

The points $q_1, q_2, q_3, \ldots$ in the figure are exactly the preimages under the embedding h_0 of the points of intersection $\alpha'_0 \cap \alpha'_1$. The hypothesis signifies

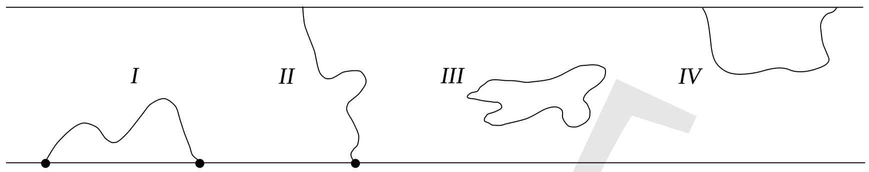


Figure 3.13:

that there exists at least one component Γ_1 of type I; we obtain Γ_0 by choosing the arc $\overline{q_1q_2}$ of $S^1 \times \{0\}$ which is homotopic to Γ_1, with endpoints fixed, in $S^1 \times [0,1]$. ∎

Lemma 3.12 $(2) \Longrightarrow (3)$

Proof. If the components $\widetilde{\alpha_0}$ and $\widetilde{\alpha}_1$ cut each other in more than one point in $\widetilde{M}$, it is easy to find an embedded disk Δ in $\widetilde{M}$ whose boundary is the union of an arc of $\widetilde{\alpha}_0$ and an arc of $\widetilde{\alpha}_1$. On Δ, one sees that $p^{-1}(\alpha_0' \cup \alpha_1')$ as in figure 3.14, where $p^{-1}(\alpha_0')$ is a dashed and $p^{-1}(\alpha_1')$ is drawn as a solid line.

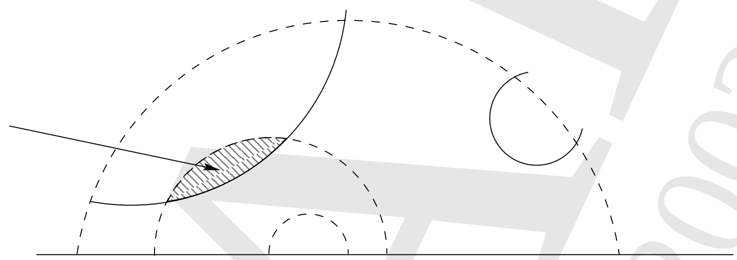


Figure 3.14:

We may find a (minimal) disk δ whose boundary is also the union of a dashed arc and a solid arc and whose interior does not meet $p^{-1}(\alpha_0' \cup \alpha_1')$. The immersion p embeds the boundary of δ because of minimality. Now, we may affirm that p embeds δ, for an immersion in codimension 0 which embeds the boundary and whose interior does not meet the border is an embedding (the number of points for the fibre is locally constant.) ∎

Hence, we have proved the equivalence of conditions (1), (2), (3) of proposition 3.10.

It remains to prove $(1) \implies (4)$. This follows immediately from proposition 3.13 and theorem 3.16.

Proposition 3.13 *Let α_0', α_0'', and α_1' be three simple curves in M, not homotopic to zero. We suppose*

1. *α_0' and α_0'' belong to the same isotopy class α_0, which is distinct from the isotopy class α_1 of α_1';*
2. $\mathrm{card}(\alpha_0' \cap \alpha_1') = \mathrm{card}(\alpha_0'' \cap \alpha_1') = i(\alpha_0, \alpha_1)$.

Then there exists an ambient isotopy of the pair (M, α_1') which pushes α_0' onto α_0''.

Extension. Using the same proof as below, the proposition remains valid if α_1' is a simple arc representing a non-trivial element of $\pi_1(M, \partial M)$.
Proof.
Let $h\colon S^1 \times [0,1] \to M$ be a transverse map onto a_1', whose restriction $h|S^1 \times \{0\}$ (respectively $h|S^1 \times \{1\}$) parameterizes α_0' (respectively α_1').

Lemma 3.14 *The closed components of $h^{-1}(\alpha_1')$ are homotopic to zero in $S^1 \times [0,1]$.*

Proof. Let γ be a component of $h^{-1}(\alpha_1')$, not homotopic to zero in $S^1 \times [0,1]$; then γ is isotopic to the boundary. Let d be the degree of $h\colon \gamma \to \alpha_1'$. We cannot have $d = 0$, since otherwise α_0' is homotopic to zero.

We cannot have $|d| > 1$, since otherwise, a nontrivial multiple of α_1' is an embedded curve, that is to say α_0'. This is known to be impossible (see Epstein [Eps66b]). If $|d| = 1$, this means α_0' is homotopic to α_1', which case we have excluded. ∎

Conclusion of proof of proposition 3.13. In the following, the components of $h^{-1}(\alpha_1')$ are of the type I, II, II, IV as in figure 3.13. In fact, by the second hypothesis, types I and IV do not exist. This is so, as $\pi_2(M, \alpha_1') = 0$, it is easy to kill the components of type III. If after this, $h^{-1}(\alpha_1')$ is empty, we conclude that α_0' and α_0'' are homotopic (hence isotopic) in $M - \alpha_1'$ and we have the conclusion of the proposition by extending the isotopy to support in $M - \alpha_1'$. If not, the rest of the components are of type II, which we may draw as vertical.

However, in general, the map h thus constructed is singular and does not give an isotopy.

Let $s_1, \ldots, s_n$ be the points of $h^{-1}(\alpha_1') \cap S^1 \times \{0\}$; these cut the circle into intervals $I_1, \ldots, I_n$ and, if $h^{-1}(\alpha_1') = \{s_1\} \times [0,1] \cup \cdots \cup \{s_n\} \times [0,1]$, we may consider $h|I_k \times [0,1]$ as a proper homotopy (i.e., the boundary moves within the boundary) between two embedded arcs in the surface N obtained by cutting M along α_1'. We remark that, by hypothesis (2), for all k, $h|I_k \times \{0\}$ represents a non-trivial element of $\pi_1(N, \partial N)$. Proposition 3.13 is then obtained by applying to each arc lemma 3.15 below, which generalizes the relative version of the result of Epstein [Eps66b]. ∎

Lemma 3.15 *Let N be a surface with boundary, and γ_0 and γ_1 two properly embedded arcs in N. Let $h\colon [0,1] \times [0,1] \to N$ be a proper homotopy between these arcs: $h(t,0)$ (respectively $h(t,1)$) parameterizes γ_0 (respectively γ_1) and $h(0,u)$ and $h(1,u)$ belong to ∂N for all u.*

Then h is deform able, rel $[0,1] \times [0,1]$, to an isotopy *from γ_0 to γ_1. Furthermore, if $h(0,u) = h(0,0)$ for all u (respectively $h(1,u) = h(1,0)$), then the deformation may be made through maps with the same properties.*

Proof. As usual these situations, the lemma is clear if γ_0 and γ_1 do not intersect except at their endpoints; indeed, γ_0 and γ_1 delimit a disc in N, in which the required isotopy is done; the isotopy is a deformation of the initial homotopy, for N is an Eilenberg-MacLane space.

In the case where they do intersect, for the separation we consider the universal covering $p\colon \widetilde{N} \to N$; consider *one* component of $p^{-1}(\gamma_0)$ and the union $\widetilde{\Gamma}_1$ of *all* components of $p^{-1}(\gamma_1)$. If we are careful enough to begin with an initial isotopy, *leaving fixed* the endpoints of γ_0, to make $\mathrm{card}(\gamma_0 \cap \gamma_1)$ as small as possible then by the equivalence $(1) \iff (2)$ in proposition 3.10, $\widetilde{\gamma}_0$ meets every component of $\widetilde{\Gamma}_1$ in at most one point.

Let $\widetilde{\gamma}_1$ be any component of $\widetilde{\Gamma}_1$; we denote by $\widetilde{\gamma}_i(0)$ and $\widetilde{\gamma}_i(1)$ the endpoints of γ_i. If $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ meet (somewhere other than at their endpoints), we have the configurations of figure 3.15. In this figure, the endpoints of the arcs belong to distinct components of $\partial\widetilde{N}$, unless explicitly indicated otherwise.

Configuration (I) is excluded; indeed, this configuration forbids the existence of a proper separating homotopy. Similarly, (II) is excluded in the case

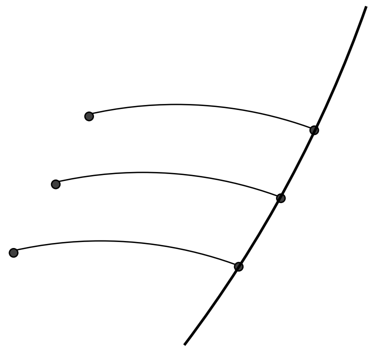


Figure 3.15:

where $h(0,u)$ is fixed. By the same argument, configurations (III) and (IV) are excluded, if in addition, $h(1,u)$ is fixed.

Thus, in the case where the endpoints are fixed, the lemma is proved.

Let us analyse the case where the origin $\widetilde{\gamma}_0(0)$ is fixed; then we only have configurations (III) and (IV). We see in $\widetilde{N}$ a triangle Δ. At the cost of changing components of $\widetilde{\gamma}_1$, we may suppose that $\mathrm{int}(\Delta) \cap \widetilde{\Gamma}_1 = \emptyset$. Therefore, $p|\Delta$ is an embedding; there is an isotopy of γ_0 supported in a neighbourhood of $p(\Delta)$, which reduces the number of intersections with γ_1 by at least one. We continue in this manner until $\mathrm{int}\,\widetilde{\gamma}_0 \cap \widetilde{\Gamma}_1 = \emptyset$. We treat similarly the case where the two endpoints are free. ∎

Theorem 3.16 *If a surface M has a metric of curvature -1, all simple curves not homotopic to zero are isotopic to a simple geodesic. Moreover, two simple geodesics meet in the minimal number of points of intersection in their isotopy classes.*

Proof. The second part of the theorem follows from the implication $(3) \Longrightarrow (1)$ in proposition 3.10.

Let $f\colon S^1 \to M$ be an embedding not homotopic to zero; by lemma 3.4, f is homotopic to a geodesic immersion g. Let $p\colon \widetilde{M} \to M$ be the universal covering. Let $\widetilde{f}_0, \widetilde{f}_1\colon \mathbb{R} \to \widetilde{M}$ be two proper embeddings with distinct images under f; let $\widetilde{g}_0$ and $\widetilde{g}_1$ be the geodesic maps which are their homotopic images.

By lemma 3.1, $\widetilde{g}_0$ and $\widetilde{g}_1$ are embeddings which have at most one point in common. This shows that $\widetilde{g}_0$ and $\widetilde{g}_1$ do not again meet.

If $\widetilde{M}$ is regarded as the interior of the Poincaré disk $\mathbb{D}^2$, then for $i = 0, 1$, $\widetilde{g}_i$ has two limit points. Since the homotopy from $\widetilde{g}_i$ to $\widetilde{f}_i$ is obtained by lifting a homotopy in M, the hyperbolic distance from $\widetilde{g}_i(x)$ to $\widetilde{f}_i(x)$ is uniformly bounded for $x \in \mathbb{R}$. In a neighbourhood of infinity, the euclidean arc length is infinitesimally small compared with the hyperbolic arc length; hence as $x \to \pm\infty$, the euclidean distance from $\widetilde{g}_i(x)$ to $\widetilde{f}_i(x)$ tends to zero. Hence $\widetilde{f}_i$ has the same limit points on $\partial\mathbb{D}^2$ as $\widetilde{g}_i$. Now, if $\widetilde{g}_0$ and $\widetilde{g}_1$ have a common point, then by an intersection homology argument (or by the Jordan theorem), $\widetilde{f}_0$ and $\widetilde{f}_1$ must meet again. This is impossible, for f is an embedding.

Thus we have proved that the image of g is a simple curve which g covers a certain number of times. To see that g as an embedding, we apply the result of Epstein cited in the beginning of the paragraph. ∎

We can give an application of the theorem which clarifies condition (3) of proposition 3.10.

Corollary 3.17 *Let α'_0 and α'_1 be two simple curves which intersect transversely. We suppose that they have components $\widetilde{\alpha}_i$ $(i = 0, 1)$ in $p^{-1}(\alpha'_i)$ in the universal covering satisfying* $\operatorname{card}(\widetilde{\alpha}_0 \cap \widetilde{\alpha}_1) = \infty$. *Then the classes α_0 and α_1 in $\mathcal{S}$ are equal.*

Proof. Considering the hypothesis of transversality, we have $\operatorname{card}(\alpha'_0 \cap \alpha'_1) < \infty$. Therefore there are points $* \in \alpha'_0 \cap \alpha'_1$ and $x, y \in \widetilde{\alpha}_0 \cap \widetilde{\alpha}_1$, such that $x \neq y$, $p(x) = p(y) = *$. We orient every arc $\widetilde{\alpha}_i$ from x to y and α'_i like $\widetilde{\alpha}'_i$. Consider α_0, α_1 as elements of $\pi_1(M, *)$. The segment from x to y on $\widetilde{\alpha}_0$ (respectively $\widetilde{\alpha}_1$) covers α'_0 k times, (respectively α'_1 l times). We therefore have in $\pi_1(M, *)$ the equality

$$\alpha_0^k = \alpha_1^l.$$

Now, give M a metric of curvature -1. If g_i designates the (unique) geodesic of $\widetilde{\mathcal{M}}$ invariant under T_{α_i}, we see that $T_{\alpha_0^k} = T_{\alpha_1^l}$ leaves invariant g_0 and g_1. Thus $g_0 = g_1$, $p(g_0) = p(g_1)$ and α'_0, α'_1 are (freely) homotopic to the same geodesic in M. ∎

From the equivalence (1) $\iff$ (2) in proposition 3.10, we deduce the following fact. Let α', β', γ' be three simple arcs $\not\simeq 0$ in M, with $\alpha' \cap \gamma' =$

$\beta' \cap \gamma' = \emptyset$; if $\mathrm{card}(\alpha' \cap \beta')$ is minimal in $M - \gamma'$, then $\mathrm{card}(\alpha' \cap \beta')$ is also minimal in M. This criterion will be used in what follows.

We recall from exposé 1 that $\mathbb{P}(\mathbb{R}_+^{\mathcal{S}})$ is the "projective" space associated to $\mathbb{R}_+^{\mathcal{S}}$ and that

$$\pi : \mathbb{R}_+^{\mathcal{S}} - \{0\} \to P(\mathbb{R}_+^{\mathcal{S}})$$

is the natural projection.

Proposition 3.18 *1. The image of i_8 is continuous in $\mathbb{R}_+^{\mathcal{S}} - \{0\}$.*

2. The map πi_* (in particular i_*) is injective.

Proof. It suffices to prove that if $\alpha_1 \neq \alpha_2 \in \mathcal{S}$, there exists $\beta \in \mathcal{S}$ such that

$$i(\alpha_1, \beta) = 0 \neq i(\alpha_2, \beta).$$

If $i(\alpha_1, \alpha_2) \neq 0$, it suffices to take $\beta = \alpha_1$. If $i(\alpha_1, \alpha_2) = 0$, there exist simple curves $\alpha_1' \in \alpha_1$ and $\alpha_2' \in \alpha_2$ such that $\alpha_1' \cap \alpha_2' = \emptyset$. By cutting m along α_1', we obtain a surface N containing α_2' in its interior.

As α_2' is not isotopic to α_1', there exists in N a curve β' not separable from α_2' in N. If α_2' does not separate N, we take β' with $\mathrm{card}(\beta' \cap \alpha_2') = 1$. If α_2' separates N into N_1 and N_2, we take $\beta' = I_1 \cup I_2$ where I_j is an arc representing a non-trivial element of $\pi_1(N_j, \alpha_2')$; this is possible since neither N_1 nor N_2 is an annulus or a disk.

If β is the isotopy class of β' in M, we have, by proposition 3.10, that $i(\alpha_2, \beta) \neq 0$. ∎

3.4 Systems of simple curves on M and hyperbolic isometries

I consider a system of distinct elements $\alpha_1, \ldots, \alpha_k \in \mathcal{S}$, with the property that $\alpha_l, \alpha_q) \leq 1$. We define the complex $\Gamma(\alpha_1, \ldots, \alpha_k)$ by taking as vertices $\alpha_1, \ldots, \alpha_k$; the vertices α_l, α_q are joined by an edge if $i(\alpha_l, \alpha_q) = 1$. I will henceforth suppose that $\Gamma(\alpha_1, \ldots, \alpha_k)$ is a tree.

Lemma 3.19 *Under the conditions above, let $\alpha_j', \alpha_j'' \in \alpha_j$ we such that $\mathrm{card}(\alpha_l', \alpha_q') =$ $\mathrm{card}(\alpha_l'', \alpha_q'') = i(\alpha_l, \alpha_q)$. Then there exists a diffeomorphism of M, isotopic to the identity, which transforms $\cup\alpha_j'$ into $\cup\alpha_j''$.*

Proof. For $k = 2$, this is proposition 3.13. To apply induction, we suppose that $\alpha'_j = \alpha''_j$ for $j \leq l$, $l \geq 2$, the indexing being compatible with the tree structure. Let p, q be such that $p \leq l < q$, and $i(\alpha_p, \alpha_q) = 1$. Let N be the manifold obtained by cutting M along the arcs α'_j, where $j \leq l, j \neq p$. Then α'_p is cut in one of many arcs in N; let I be one such which meets α'_q (α'_q is a closed curve in N since Γ is a tree); as $\mathrm{card}(\alpha'_q \cap I) = 1$, the arc I represents a non-trivial element of $\pi_1(N, \partial N)$.

We prove that α_q cuts in one point the same arc I (and not some possibly different component of $\alpha'_p \cap N$). If not, for some $j \neq p$ such that $j \leq l$ and $i(\alpha_j, \alpha_p) = 1$, we have $\alpha_j = \alpha_q$ [look at the preimage of α'_j of the range of the homotopy from α'_q to α''_q in M; one of these components is necessarily parallel to the boundary of an annulus]. The extension of proposition 3.13 is now applicable: we have, in N, an isotopy which pushes α''_q onto α'_q and which leaves $\alpha'_p \cap N$ alone. ∎

Application. Let ρ be a metric of curvature -1 on the surface M. We consider the simple curves $\alpha'_1, \ldots, \alpha'_k$ as in figure 3.16 (we consider M to be closed here); $M - \cup\alpha'_j$ is a cell. Let α''_j be the geodesic, in the metric ρ, in the isotopy class of α'_j; we may check that $\mathrm{card}(\alpha''_l \cap \alpha''_q) = \mathrm{card}(\alpha'_l \cap \alpha'_q)$. By lemma 3.19, $M - \cup\alpha''_j$ is a cell. In particular, figure 3.16 is realizable in geodesics.

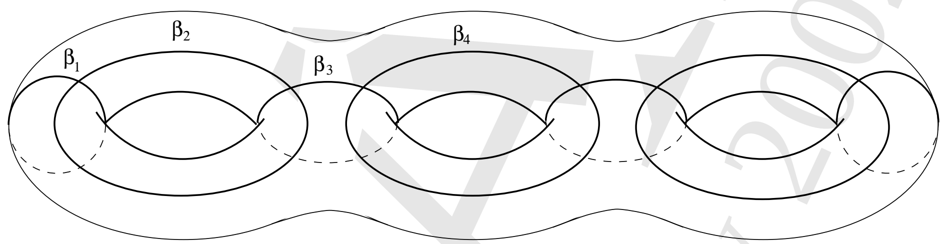


Figure 3.16:

Theorem 3.20 *Let ρ be a metric of curvature -1 on a compact surface M. The group $I(M, \rho)$ of ρ-isometries is* finite *and the only isometry isotopic to the identity is the identity.*

Proof. I begin by considering the set M^M of *all* maps $M \to M$, with the topology of pointwise convergence. By the theorem of Tychonov, M^M is compact.

We remark, as an aside, that on $I(M, \rho)$ the topology of pointwise convergence and the uniform topology coincide. [Indeed, if I consider a *finite* set X in M, sufficiently dense, an isometry is completely characterized by what it does on X ....] We remark that $I(M, \rho)$ is closed in M^M.

Furthermore, I claim that an isometry isotopic to the identity is equal to the identity. Indeed, let ϕ be such an isometry; the action of ϕ on $\mathcal{S}$ is trivial; by the unicity of geodesics in a class $\alpha \in \mathcal{S}$ in hyperbolic geometry, the geodesic g_α in the class $\alpha \in \mathcal{S}$ is invariant: $\phi(g_\alpha) = g_\alpha$. We deduce easily that ϕ is the identity on the system of geodesics in figure 3.16. Hence, ϕ is the identity on the complementary cell.

Thus, $I(M, \rho)$ is discrete. But a closed discrete set in a compact space is finite. ∎

Corollary 3.21 *Let $f \in \mathrm{Diff(M)}$ and let $\mathcal{T}(f)$ be the natural action of f on the Teichmüller space of M (see exposé 7). If $\mathcal{T}(f)$ has a fixed point, there is a periodic diffeomorphism of M isotopic to f.*

Chapter 4

The space of simple closed curves on a surface

by V. Poénaru

4.1 The weak topology

Let M be a closed oriented surface of genus $g \geq 2$. Denote by $\mathcal{S}$ the space of isotopy (= homotopy) classes of simple, closed, not oriented curves which are not homotopic to zero in M. We have already seen (exposé 3, section 3) that the composite map

$$\mathcal{S} \xrightarrow{i_*} \mathbb{R}_+^{\mathcal{S}} - \{0\} \xrightarrow{\pi} P(\mathbb{R}_+^{\mathcal{S}})$$

is *injective*. The map i_* extends to a map which we will denote by the same symbol:

$$i_* \colon \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}}$$

by the formula

$$i_*(\lambda, \alpha)(\beta) = \lambda i(\alpha, \beta) \qquad \text{where } \lambda \in \mathbb{R}_+, \text{ and } \alpha, \beta \in \mathcal{S}$$

Remark. If $\overline{i_*(\mathbb{R}_+ \times \mathcal{S})}$ designates the closure of $i_*(\mathbb{R}_+ \times \mathcal{S})$ in $\mathbb{R}_+^{\mathcal{S}}$, then

$$\pi(\overline{i_*(\mathbb{R}_+ \times \mathcal{S})} - \{0\}) = \overline{\pi i_*(\mathcal{S})}.$$

This is a general fact about cones.

Proposition 4.1 *In $P(\mathbb{R}_+^{\mathcal{S}})$, the set $\pi i_*(\mathcal{S})$ is relatively compact.*

For the proof, we begin by choosing for M a metric ρ of curvature -1, and denote by $\ell(\alpha)$ the ρ-length of the unique geodesic belonging to the class of $\alpha \in \mathcal{S}$.

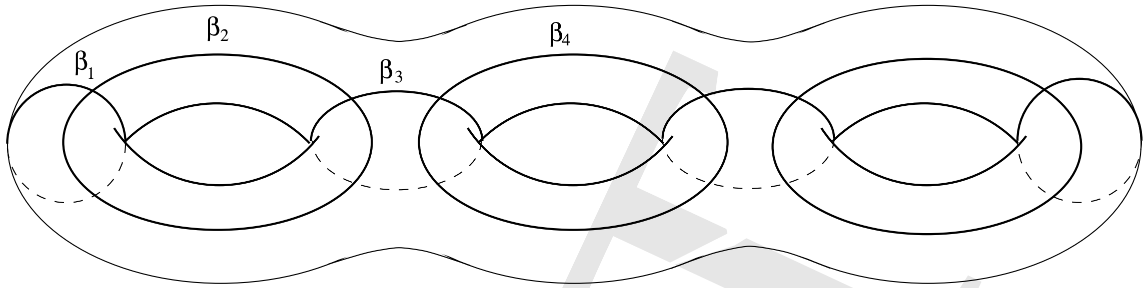


Figure 4.1:

Lemma 4.2 *There exists a constant $C = C(M, \rho)$ such that for all $\alpha, \beta \in \mathcal{S}$*

$$i(\alpha, \beta) \leq C \cdot \ell(\alpha) \cdot \ell(\beta).$$

Proof. If $\alpha = \beta$, we have $i(\alpha, \beta) = 0$ and the inequality is clear. Suppose therefore that $\alpha \neq \beta$. Let ϵ be a positive number smaller than the radius of injectivity of the exponential map. The geodesic g_α in the isotopy class α may be covered by fewer than $(\frac{\ell(\alpha)}{\epsilon} + 1)$ short arcs each contained in a geodesic disk. The same holds for g_β. A small arc of g_α cuts in at least one point a small arc of g_β, by the definition of radius of injectivity: therefore, in a small arc of g_α, there are at least $(\frac{\ell(\beta)}{\epsilon} + 1)$ points of intersection with g_β. We therefore find

$$i(\alpha, \beta) = \operatorname{card}(g_\alpha \cap g_\beta) \leq \left(\frac{\ell(\alpha)}{\epsilon} + 1\right)\left(\frac{\ell(\beta)}{\epsilon} + 1\right)$$

As $\ell(\alpha) > \epsilon$, the desired inequality is clear. ∎

On M, we now consider the system of elements $\beta_1, \ldots, \beta_{2g+1} \in \mathcal{S}$ represented in figure 4.1. In section 3.4, we saw that such a system may be realized by geodesics $g_{\beta_1}, \ldots, g_{\beta_{2g+1}}$.

Lemma 4.3 *There exists a constant c such that for all $\alpha \in \mathcal{S}$,*

$$\sum_j i(\alpha, \beta_j) \geq c \cdot \ell(\alpha)$$

Proof. The system $\{g_{\beta_j}\}$ decomposes M in a number of simply connected regions. In each, the length of a geodesic arc is bounded, say by L. Thus, we have the desired result by taking $c = 1/L$. ∎

Proof.[of Proposition 4.1] For a fixed constant C, consider the subset $S(C) \subset P(\mathbb{R}_+^{\mathcal{S}})$, defined by

$$S(C) := \left\{ f \in P(\mathbb{R}_+^{\mathcal{S}}) \mid \forall \beta \in \mathcal{S}, f(\beta) \leq C \cdot \ell(\beta) \right\}$$

By Tychonov's theorem, $S(C)$ is compact. Now taking C to be the constant of lemma 4.2, consider $S_0 \subset S(C)$, which is elements in $\mathbb{R}_+^{\mathcal{S}}$ of the set of functionals of type $i_*(\alpha)/\ell(\alpha)$. By lemma 4.3, we see that $S_0 \subset \mathbb{R}_+^{\mathcal{S}} \setminus \{0\}$. On the other hand, S_0 is compact, thus $\pi(S_0)$ is compact. By lemma 4.3, we have $\pi i_*(\mathcal{S}) \subset \pi(S_0)$; this gives the compactness of $\overline{\pi i_*(\mathcal{S})}$. ∎

4.2 The space $\mathcal{S}'$ of multiple curves

As $\mathcal{S}$ is difficult to study, we introduce a space which is larger, and easier to study. Let $\mathcal{S}' = \mathcal{S}'(M)$ be the space of isotopy classes of closed submanifolds of dimension 1 (not oriented and not necessarily connected) none of whose components are homotopic to zero. As in the case of simple curves, we define $i(\alpha, \beta)$ for $\alpha \in \mathcal{S}'$ and $\beta \in \mathcal{S}$, as well as $i_* \colon \mathcal{S}' \to \mathbb{R}_+^{\mathcal{S}}$ and $\pi i_* \colon \mathcal{S}' \to P(\mathbb{R}_+^{\mathcal{S}})$. The minimal intersection between a multiple curve and a simple curve is the sum of the minimal intersections of the different components.

Remark: By the same reasoning as in section 3.3, we may prove that i_* is injective and that two elements α_1 and α_2 of $\mathcal{S}'$ are the same image under πi_* if and only if they are integer multiples of the same $\alpha_0 \in \mathcal{S}'$. [That is, one has a natural map $\mathbb{N} \times \mathcal{S}' \to \mathcal{S}'$.]

Theorem 4.4 *In $P(\mathbb{R}_+^{\mathcal{S}})$, we have*

$$\overline{\pi i_*(\mathcal{S})} = \overline{\pi i_*(\mathcal{S}')}$$

By an application of [???] proposition 4.1, we obtain

Corollary 4.5 *In $\mathbb{R}_+^{\mathcal{S}}$, we have*

$$i_*(\mathcal{S}') \subset i_*(\mathbb{R}_+ \times \mathcal{S})$$

Proof.[of theorem 4.4] The point is to show that $\pi i_*(\mathcal{S})$ is dense in $\pi i_*(\mathcal{S}')$. Let $\alpha \in \mathcal{S}'$ be represented by a union of pairwise disjoint simple curves $\alpha_1, \dots, \alpha_k$. We may choose a simple, connected curve γ such that $\mathrm{card}(\gamma \cap \alpha_j)$ is equal to $i(\gamma, \alpha_j)$ and non-zero for all j. Let $n_1, \dots, n_k$ be positive integers. We shall construct an element $\Gamma(n_1, \dots, n_k)$ of $\mathcal{S}$. Each arc of γ which crosses a small tubular neighbourhood of α_j is replaced by an arc with the same endpoints making n_j positive turns (see figure 4.2 for the case $n_j = 2$.) We obtain by this construction a curve $\Gamma(n_1, \dots, n_k)$, well-defined up to isotopy. We prove in the appendix to this chapter that for $\beta \in \mathcal{S}$, we have the inequality

$$\left| i(\Gamma(n_1, \dots, n_k), \beta) - \sum_j n_j \cdot i(\gamma, \alpha_j) \cdot i(\alpha_j, \beta) \right| \leq i(\gamma, \beta)$$

Take $n_j = n \prod_{\ell \neq j} i(\gamma, \alpha)_\ell)$. We obtain a curve which we'll denote by $\Gamma(n)$; it follows that

$$\left| i(\Gamma(n), \beta) - n \sum_j i(\gamma, \alpha_j) \Big[\sum_j i(\alpha_j, \beta) \Big] \right| \leq i(\gamma, \beta)$$

That is, when we projectivize and as n tends to infinity, the contributions of γ to the intersection become negligible. Thus all of $\pi i_*(\Gamma(n))$ tends to $\pi i_*(\alpha)$. ∎

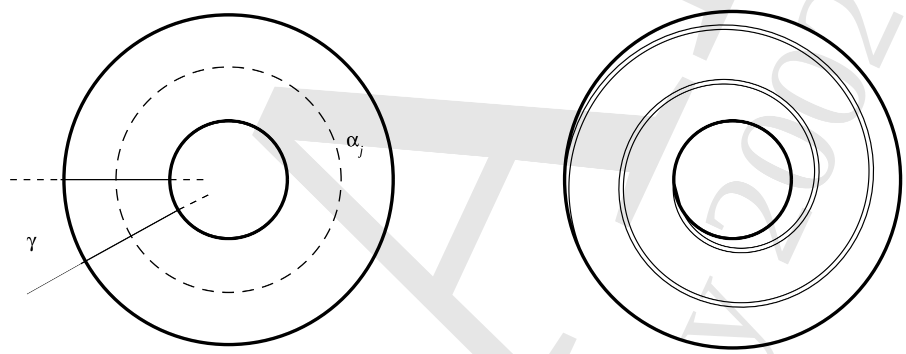


Figure 4.2:

4.3 An explicit parameterization of the space of multiple curves

Recall that P^2 denotes the standard pair-of-pants; the boundary curves are labeled $\partial_1 P^2$, $\partial_2 P^2$, $\partial_3 P^2$. In section 2.3, we classified the multiple arcs of P^2.

An element τ of $A'(P^2)$, the space of multiple arcs, is completely characterized by the three integers $m_j = i(\tau, \partial_j P^2)$ for $(j = 1, 2, 3)$; a triple of integers, not all zero, describes a multiple arc exactly when $m_1 + m_2 + m_3$ is even.

In every class of $A'(P^2)$, we choose once-and-for-all a **canonical representative**, as shown in figure 4.3. For each $\tau \in A'(P^2)$ and each $\partial_j P^2$, we choose an arc x_j from the connected components of $\partial_j P^2 - \tau$, as in figure 4.3. This choice is uniquely defined, since (P^2, τ) does not admit any non-trivial orientation-preserving automorphisms.

canonical representative

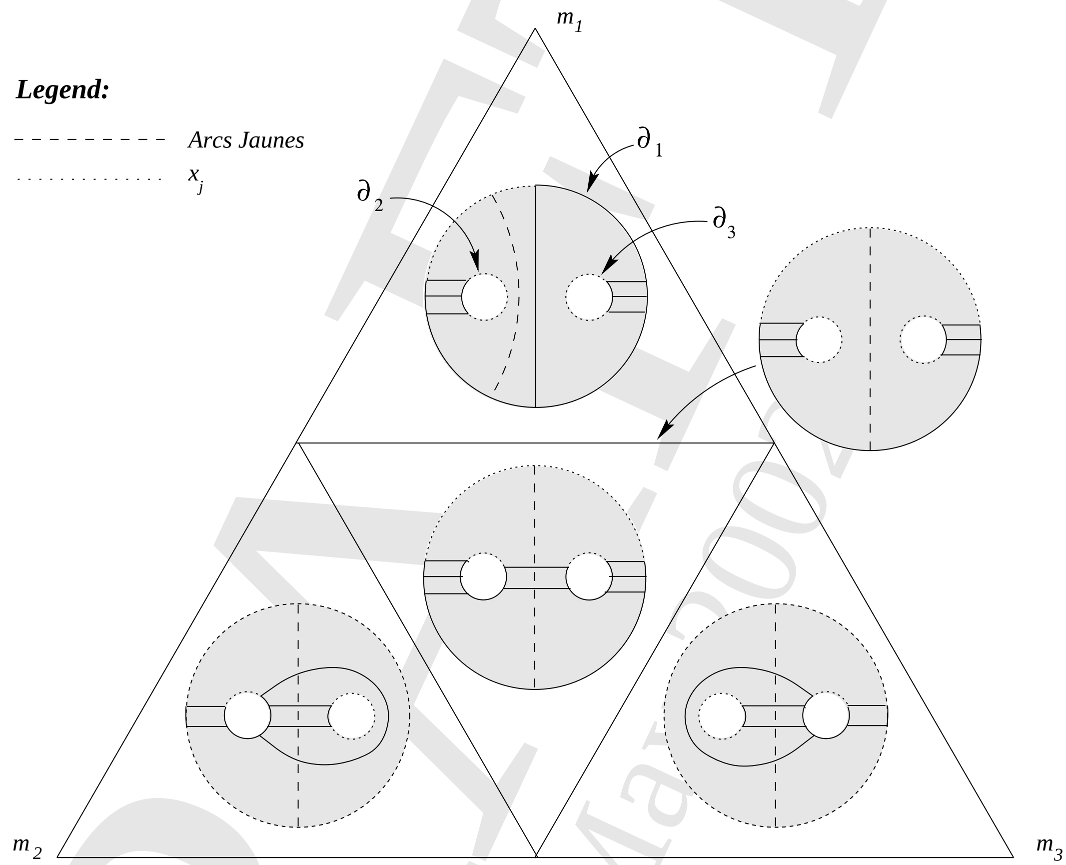


Figure 4.3:

For each model τ, we chose an *"arc jaune"* $J_1 = J_1(\tau)$ which has the following properties:

1. J_1 is a simple arc joining $\partial_1 P^2$ to itself and which cuts P^2 into two regions, one of which contains $\partial_2 P^2$, the other $\partial_3 P^2$;

2. J_1 has one endpoint in the arc $x_1(\tau)$.

3. J_1 has a minimal intersection with τ.

Similarly, we construct arcs J_2 and J_3.

Remark: In chapter 6, we classify the measured foliations on P^2. The models of figure 4.3 are the "discrete models" for these foliations, where we do not see any non-singular leaves. Moreover, for the classification of multiple curves, we follow a procedure analogous to that which we will follow in the classification of measured foliations: for example the technique of the arc jaunes which [???] we set to locate the work of gluing the pants together to reconstruct the surface.

To parameterize $\mathcal{S}'$, we make a number of choices.

1. We choose $3g - 3$ simple curves $K_1, K_2, \ldots, K_{3g-3}$, mutually disjoint, cutting M into $2g - 2$ regions diffeomorphic to pants. In addition, we take these K_i to be connected in M; in this way, the pairs-of-pants R_j are embedded in M, that is, each K_i belongs to **two** distinct pairs of pants.

2. For each K_j, we choose two simple curves K'_j and K''_j as in figure 4.4 (this is possible because of the previous condition); K'_j and K''_j differ by a positive Dehn twist along K_j (this does not depend on the orientation of the surface nor that of K_j).

3. We give each K_j a tubular neighbourhood $K_j \times [-1, 1]$; these are taken to be pairwise disjoint; the complement of their union is a number of pairwise disjoint pairs-of-pants $R'_1, R'_2, \ldots, R'_{2g-2}$.

4. Each R'_j is parameterized by P^2, via diffeomorphisms ϕ_j, fixed, but only up to isotopy.

We consider in $\mathbb{R}_+^{9g-9}$ the cone

$$B := \{(m_i, s_i, t_i) \mid i - 1, \ldots, 3g - 3;\ m_i, s_i, t_i \geq 0,\ (m_i, s_i, t_i) \in \partial(\leq \nabla)\}.$$

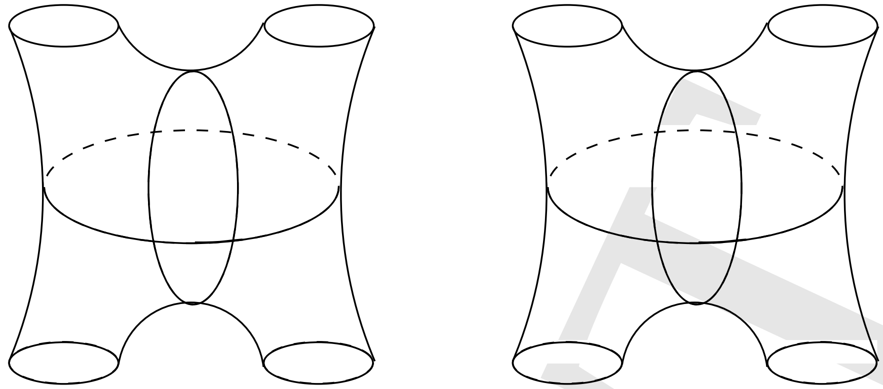


Figure 4.4:

It is homeomorphic to $\mathbb{R}^{6g-6}$ (the cone on $\partial(\leq \nabla)$ is homeomorphic to $\mathbb{R}^2$). We will construct a classification map $\Phi\colon \mathcal{S}' \to B$.

Let $\beta \in \mathcal{S}'$; we start by defining $m_j(\beta) := i(\beta, K_j)$. Knowledge of these integers determines the models of the pairs-of-pants R'_k: the corresponding model for P^2 is carried by the diffeomorphism ϕ_k. If the representative β_0 of β is chosen with minimal intersection with the boundary of all of the pants R'_k, then $\beta_0|R'_k$ is isotopic to the model. We therefore choose β_0 *equal to the model* in all of the pairs of pants R'_k; we say that this representative is in **normal form**. Note that if β_0 has a component isotopic to K_j, this is contained in the annulus $K_j \times [-1, 1]$. normal form

Lemma 4.6 *The normal form of β is "unique". Precisely, if β_0 and β_1 are two representatives of β in normal form, then, for all $j = 1, \ldots, 2g-2$, $\beta_0 \cap K_j \times [-1, 1]$ and $\beta_1 \cap K_j \times [-1, 1]$ are isotopic relative to the boundary.*

Proof. We have need for an extension of proposition 3.13 to the case that one of the curves is a multiple curve; the proof is analogous. The point is the following: if γ_0 is a component of β_0 and if γ_1 is the corresponding component of β_1, then there exists an isotopy of M which pushes γ_0 onto γ_1 and which leaves invariant all of the curves $K_j \times \{-1\}$ and $K_j \times \{+1\}$, $j = 1, \ldots, 3g-3$. In reality, it does not hurt the proof of the cited proposition that the [??] rest is as follows: if $\gamma_0 \cap K_j \times \{\pm 1\} = \gamma_1 \cap K_j \times \{\pm 1\} = \emptyset$, then γ_0 is isotopic to γ_1 in $M - K_j \times \{\pm 1\}$. This assertion is true for the "classical" arguments (lemma 3.14) except possibly if γ_0 is isotopic to K_j. but then, because of the "normal form" condition, there is nothing to prove.

This being so, in the discussion below, we may replace γ_0 (resp. γ_1) by the packet $\bar{\gamma}_0$ (resp. $\bar{\gamma}_0$) of all of the components of β_0 (resp. β_1) parallel to γ_0 (resp. γ_1). We may thus fabricate a normal form β_0' with the following properties:

1. β_0' and β_0 are isotopic by an isotopy which respects the curves $K_j \times \{\pm 1\}$.
2. The packet $\bar{\gamma}_0'$, corresponding to $\bar{\gamma}_0$, coincides with $\bar{\gamma}_1$.

Now let δ_0 be a curve in $\beta_0' - \bar{\gamma}_1$ and let δ_1 be the corresponding curve in [???] $\beta_1 - \bar{\gamma}_1$. If δ_0 is not parallel to $\bar{\gamma}_1$, δ_0 and δ_1 are isotopic in $M - \bar{\gamma}_1$. Always by the same arguments, we find then that there exists an isotopy of M, constant on $\bar{\gamma}_1$ and respecting the curves $K_j \times \{\pm 1\}$ which pushes δ_0 onto δ_1. We continue in this way with the rest. In the end, β_0 and β_1 are isotopic, by an isotopy which respects all of the curves $K_j \times \{\pm 1\}$.

Now, we claim that the isotopy above may be chosen to be constant in all of the small pairs of pants R_j', which will prove the lemma. It is clear that $\beta_0 \cap R_j'$ is empty. Otherwise, it is due to the fact that the loops of $\mathrm{Diff}(P^2, \partial_1, \partial_2, \partial_3)$ are all homotopic to zero (see chapter 1.6). ∎

The lemma is essential for the pursuit of the classification. The models $\beta_0 \cap R_\ell'$ are equipped with their "arcs jaunes". Consider the curve K_j and the two adjacent pants R_1 and R_2. In the small pairs-of-pants R_1' and R_2', we have the two arcs jaunes J_1 and J_2 emanating from the respective boundaries parallel to K_j. There exists in $K_j \times [-1, 1]$ simple arcs S_j, S_j', T_j, T_j' such that $J_1 \cup S_j \cup J_2 \cup S_j'$ is isotopic to K_j' and $J_1 \cup T_j \cup J_2 \cup T_j'$ is isotopic to K_j''. If we impose the condition that $\partial S_J = \partial T_J$ and $\partial S_J' = \partial T_J'$, $S_j \cap S_j' = \emptyset$, $T_j \cap T_j' = \emptyset$, then $S_j \cup S_j'$ (resp. $T_j \cup T_j'$) is unique up to isotopy relative to the boundary. On the other hand, $T_j \cup T_j'$ is obtained from $S_j \cup S_j'$ by a positive Dehn twist in the annulus.

Then, if the endpoints of the arcs are not in β_0, this is a way to put the arcs into a position of minimal intersection with β_0. This being done, we put:

$$\begin{aligned} s_j(\beta) &:= \mathrm{card}(\beta_0 \cap S_j) \\ t_j(\beta) &:= \mathrm{card}(\beta_0 \cap T_j) \end{aligned}$$

Lemma 4.7 *For each j, the triple $(m_j(\beta), s_j(\beta), t_j(\beta))$ is an element of the boundary $\partial(\leq \nabla)$ of the triangle inequality. [Compare with the classification theorem for $\mathcal{S}'(T^2)$ in chapter .]*

Proof. The proof is shown in figure 4.5. ∎

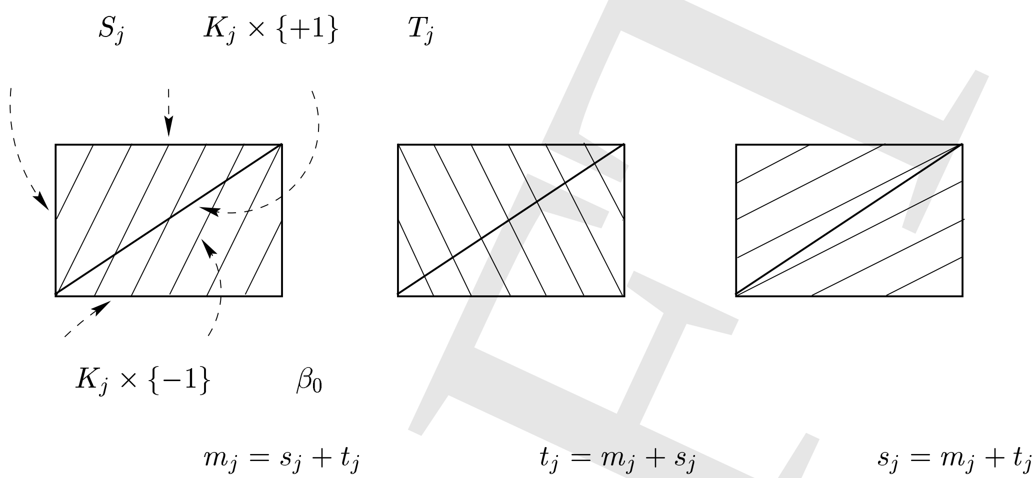


Figure 4.5: The annulus $K_j \times [-1, 1]$ is cut along S_j.

Let $B_0 \subset B$ be the set of points $\neq 0$, with integer coordinates, satisfying the following additional condition: if K_{j_1}, K_{j_2}, K_{j_3}, are on the boundary of the same pair of pants, then $m_{j_1} + m_{j_2} + m_{j_3}$ is even.

Theorem 4.8 *The map $\Phi \colon \mathcal{S}' \to B$ is a bijection of $\mathcal{S}'$ with B.*

Remark: By an analogous procedure, we will classify the measured foliations and Teichüller structures. In reality, as we will explain, theorem 4.8 above is strictly contained within the classification theorem for measured foliations. But the simplicity of the means in this proof [???] leads us to include this particular case. [In particular, for foliations, one does obtains uniqueness of the normal form only after a long detour.]

Proof. The image is evidently contained in B_0. On the other hand, we have a recipe to make a multiple curve β from the element $\{m_j, s_j, t_j \mid j = 1, \ldots, 3g-3\}$ of B_0. As in chapter 1.6, the coefficients m_j determine the arcs in the small pants R'_k. Having these, we have also the "arcs jaunes" and hence, for each j, we get the arcs S_j and T_j in the annulus $K_j \times [-1, +1]$.

If $m_j = 0$, $s_j = t_j$ indicates the number of curves of β parallel to K_j. If $m_j \neq 0$, we already have m_j points on $K_j \times \{-1\}$ and on $K_j \times \{+1\}$; the coefficients s_j and t_j determine completely the way in which these are joined. It remains to verify that the multiple curve constructed in this way has the property of having minimal intersection with each K_j, that is to say, that $i(\beta, K_j) = m_j$; for this we use the criterion of proposition 3.10.

As soon as S_j and T_j are fixed, $\beta_0 \cap K_j \times [-1, +1]$ is determined, up to isotopy relative to the boundary, by s_j and t_j. The injectivity of Φ follows. ∎

Remark: The members of the seminar do not know how to detect which are the coefficients of a simple curve.

Obviously, Φ is homogeneous (of degree 1) by [???] relation to multiplication by an integer scalar. We may thus extend Φ by homogeneity, to $\Phi\colon \mathbb{R}_+ \times \mathcal{S} \to B$.

Corollary 4.9 *The map $\Phi\colon \mathbb{R}_+^* \times \mathcal{S} \to B$ is injective.*

Proof. If not, there exists α_0 and $\alpha_1 \in \mathcal{S}$ and a scalar $\lambda > 0$ such that $\Phi(\alpha_0) = \lambda\Phi(\alpha_1)$. It is very easy to see that λ is rational. Thus, we have integers n_0 and n_1 such that $\Phi(n_0\alpha_0) = \Phi(n_1\alpha_1)$. By theorem 4.8, we have $n_0\alpha_0 = n_1\alpha_1$. It follows immediately that $\alpha_0 = \alpha_1$. ∎

Problem: Show directly that $\Phi(\mathbb{R}_+ \times \mathcal{S})$ is dense in B. This is plausible since the (positive) cone on B_0 is dense in B. Of course, this is the result of the following theorem which is the "discrete" version of the theorem on foliations and which will not be proved until chapter 5.3.2.

Theorem 4.10 *There exists a closed cone $\mathcal{C}$ in $\mathbb{R}_+^{\mathcal{S}}$ and a continuous, positive map $\theta_C\colon \mathcal{C} \to B$, homogeneous of degree 1, which makes the following diagram commute:*

$$\begin{tikzcd}
\mathbb{R}_+ \times \mathcal{S}\textit{(resp. }\mathcal{S}') \arrow[rr, "i_*"] \arrow[dr, "\Phi"'] & & \mathcal{C} \subset \mathbb{R}_+^{\mathcal{S}} \arrow[dl, "\theta_C"] \\
& B &
\end{tikzcd}$$

Furthermore, θ_C induces a homeomorphism of $i_*(\mathbb{R}_+ \times \mathcal{S}')$ with B.

Consequences:

1. $\Phi(\mathbb{R}_+ \times \mathcal{S})$ is dense in B. (use theorem 4.4 and the fact that $\Phi(\mathcal{S}')$ is a "network" as well as the continuity and the homogeneity of θ_C.)

2. The space $\overline{\pi i_*(\mathcal{S})}$ is homeomorphic to S^{6g-7}.

Remark: The existence of θ_C signifies that the coefficients of $s_j(\beta)$ and $t_j(\beta)$ are given by continuous formulas homogeneous of degree 1 as functions of [???] $i(\beta,\alpha), \alpha \in \mathcal{S}$. We give these formulas explicitly in the framework of measured foliations; they permit the continuous interpolation of the variables.

On the other hand, as Φ is injective for all $\alpha \in \mathcal{S}$, there exists a map $\psi_\alpha\colon B_0 \to \mathbb{N}$ such that for all $\beta \in \mathcal{S}'$, we have:

$$i(\beta,\alpha) = \psi_\alpha(\Phi(\beta))$$

It appears very difficult to make these formulas explicit.

Chapter 5
Measured Foliations

by A. Fathi and F. Laudenbach

5.1 Measured foliations, the Poincaré recurrence theorem and the Euler-Poincaré formula.

5.1.1 Definition.

transverse invariant measure

Let M be a surface[1] and $\mathcal{F}$ a foliation of M with isolated singularities. We call a measure μ defined on arcs transverse to $\mathcal{F}$ a **transverse invariant measure** if it satisfies the following property:

> If $\alpha, \beta\colon [0,1] \to M$ are two arcs transverse to $\mathcal{F}$, isotopic through transverse arcs whose endpoints remain in the same leaf, then $\mu(\alpha) = \mu(\beta)$.

If the arc passes through a singularity, transversality is understood at each point where the arc belongs to a regular leaf.

N.B: In the following, we limit ourselves to the case where the measure is regular with respect to Lebesgue measure: every regular point admits a smooth chart (x, y) where the foliation is defined by dy and the measure on each transverse arc is induced by dy.

[1]The theory may be extended to non-orientable surfaces. For simplicity, we will suppose M to be orientable.

5.1.2 Singularities permitted in the interior of M.

For each integer $k > 1$, consider singularities of the form of the quadratic holomorphic differential $z^k\,dz^2$. We will consider

$$\mathrm{Im}(\sqrt{z^k\,dz^2}) = r^{k/2}\left(r\cos\left(\frac{2+k}{2}\theta\right)d\theta + \sin\left(\frac{2+k}{2}\theta\right)dr\right)$$

which is a form of degree 1, well-defined up to sign. This therefore defines a measured foliation whose origin is an isolated singularity, admitting as separatrices the half-lines $r \geq 0, \dfrac{2+k}{2}\theta = 0 \bmod \pi$.

As models for the singularity we choose a compact domain containing the origin, delimited by arcs transverse to the foliation (**faces**) and arcs contained among the leaves of $\mathcal{F}$ (**sides**). faces sides

Remark. Let ω be a closed differential form of degree 1 on M, ($\partial M = \emptyset$), whose singularities are "Morse" (i.e. generic). Suppose in addition that ω does not have a center (critical point of index 0 or 2); then ω defines a measured foliation. If it easy to see that a measured foliation is defined by a closed form if and only if it is transversely orientable in the complement of the singularities.

5.1.3 Singularities permitted in the boundary of M.

The regular points of the boundary are those where the boundary is transverse to the foliation or else those with a neighbourhood in the boundary which coincides with a leaf.

The singular points admit a chart of the form shown in 5.1 [???] traced on the upper half-plane if k is even or on the half-plane $\{z \mid \mathrm{Re}(z) < 0\}$ if k is odd.

Finally, in this entire work, given a measured foliation $(\mathcal{F}, \mu)$ on the manifold M, each point of M admits a neighbourhood with a chart foliated isomorphically with one of the models in figure 5.1.

N.B: In the chart of a singular point, we will agree that the separatrices belong to different patches. In this way, in M, each leaf is diffeomorphic to an interval of R or to S^1.

Figure 5.1:

5.1.4 Good Atlas.

If M is compact, there exists a constant ϵ_0 and two closed covers $\{U_j\}_{j\in J}$, $\{V_j\}_{j\in J}$, by domains of charts, satisfying:

1. $M = \bigcup_{j\in J}(\operatorname{int} U_j)$;

2. For each $j \in J$, $U_j \subset V_j$ and the faces of U_j are contained in the faces of V_j (see figure 5.2).

Figure 5.2:

3. Every point in a side of U_j is a distance at most ϵ_0 from the sides of V_j (all distances are measured along trajectories using the invariant measure μ.)

4. The charts U_j do not contain any singular points.

5. The intersection of two charts U_{j_1} and U_{j_2} (resp. V_{j_1} and V_{j_2}) is a rectan-

gle:

To satisfy the last condition, we chose a line-field transverse to the foliation on the complement of the singularities, and we require that the charts be sufficiently small and their faces be tangential to the line-field.

5.1.5 The Poincaré recurrence theorem.

Theorem 5.1 (Poincaré) *Let M be a compact surface equipped with a measured foliation $(\mathcal{F}, \mu)$. Let α be an arc ($\cong [0,1]$) on ∂M, transverse to $\mathcal{F}$ at all points of* $\operatorname{int}(\alpha)$*, and let x be one of its endpoints. Then the leaf L_x issuing from x goes either to a singular point or to the boundary ∂M.*

Proof. We will use the atlas described in 5.1.4. Suppose that L_x does not end in a singularity, and truncate α so that $\mu(\alpha) = \epsilon < \epsilon_0$, and that, for every $y \in \alpha$, the leaf L_y does not end in a singularity. We claim that if L_x does not meet the boundary again, then we have an injective immersion $\Phi\colon \alpha \times \mathbb{R}_+ \to M$, where $\Phi(y \times \mathbb{R}_+) = L_y$ for each $y \in \alpha$.

In effect, if P is a patch of L_x in U_i, it is in the boundary of a band of V_i, of width ϵ, which does not contain any singularities, by the hypotheses on α. If two patches of L_x meet [???] the bands in question glue together by the properties of an atlas. This gives an immersion. Injectivity holds because $\Phi^{-1}(\alpha) = \alpha \times \{0\}$ and for every point of the image of Φ, this does not pass through a leaf.

Let z be a point of recurrence of the leaf L_x; if $z \in U_i$ there exist infinitely many bands of size ϵ, components of image$(\Phi) \cap V_i$. But two distinct bands are disjoint. Impossible. ▮

Corollary 5.2 *If a leaf L of $\mathcal{F}$ is not closed in $M -$ sing(F), and if α is an arc transverse to $\mathcal{F}$ cutting L, then $\alpha \cap$ Ł is infinite.*

Proof. It suffices to show that it is impossible that $\alpha \cap L$ be an endpoint of α. For this, we cut M along $\operatorname{int} \alpha$ to obtain M', equipped with an induced foliation $\mathcal{F}'$. If C is the curve of $\partial M'$ arising from α, then $\mathcal{F}'$ has the configuration shown in 5.3; with the two singularities s_1 and s_2 corresponding to the endpoint of α, the two leaves L_g and L_d in $\mathcal{F}'$ issuing from s_1 correspond to L.

Figure 5.3:

By the theorem, L_g (resp. L_d) ends in a singularity of $\mathcal{F}'$ or on the boundary of M'. If the boundary is C, by the hypotheses on α, we conclude that $L_d = L_g$, which implies that L is closed (contradiction). If not, considering M' contained in M, L_g and L_d end in a singularity of $\mathcal{F}$ or on the boundary of M, hence L is closed (contradiction). ∎

5.1.6 The Euler-Poincaré formula.

Let M be a compact surface equipped with a foliation $\mathcal{F}$ having singularities of the type admitted in figures 5.2, 5.3. We recall that each component of the boundary is either

(A) transverse to $\mathcal{F}$, or

(B) a cycle of leaves (i.e., a finite union of leaves and singular points).

To each singularity s, we associate an integer P_s

$$P_s = \begin{cases} \text{number of separatrices} & \text{if } s \in \operatorname{int} M \text{ or if } s \in \partial M, \text{ case (B)} \\ \text{number of separatrices} + 1 & \text{if } s \in \partial M, \text{ case (A)} \end{cases}$$

Theorem 5.3 (Euler-Poincaré formula)

$$2\chi(M) = \sum_{\text{sing}(\mathcal{F})} (2 - P_s)$$

Proof. We reduce to the case where ∂M does not have any singularities, following the procedure of figure 5.4. In pushing each singularity from the boundary into the interior in this way, we may conserve the integer P_s assigned to the singularity by the rules above.

Figure 5.4:

Denote by Σ' the set of singular points with an odd number of separatrices, and by Σ'' the set of singular points with an even number of separatrices; let $\Sigma = \Sigma' \cup \Sigma''$. We have an orientation homomorphism of the fibre tangent to $\mathcal{F}$:

$$\pi_1(M - \Sigma) \to \mathbb{Z}/2.$$

This defines a 2-sheeted covering which prolongs above Σ'' and is branched over Σ'. We therefore have a branched covering $p\colon \widetilde{M} \to M$, where $\widetilde{M}$ is equipped with an orientable singular foliation $\widetilde{\mathcal{F}}$, which we may think of as generated by a vector field $\widetilde{X}$; if s is a singularity of $\widetilde{\mathcal{F}}$, then P_s is an even integer and the index of $\widetilde{X}$ in s is $-\dfrac{P_s}{2} + 1$. Then, if there is not a singularity on the boundary, we have

$$\chi(\widetilde{M}) = \sum_{\text{sing}(\widetilde{X})} \text{indices} = \sum_{\text{sing}(\widetilde{\mathcal{F}})} \frac{P_s}{2} + 1$$

but

$$\chi(\widetilde{M}) = 2\chi(M) - \text{card}(\Sigma') \qquad \text{and} \qquad \sum_{\text{sing}(\widetilde{\mathcal{F}})} 1 = 2 \cdot \text{card}(\Sigma'') + \text{card}(\Sigma').$$

Thus, if $p(s) \in \Sigma''$, $P_s = P_{p(s)}$, but s has a "twin"; if $p(s) \in \Sigma'$, $P_s = 2P_{p(s)}$. Rearranging the equalities gives the desired formula. ∎

N.B. In computations, we must not forget that $P_s \geq 3$.

5.1.7 Quasi-transverse curves.

quasi-transverse

We say that a curve γ is **quasi-transverse** to $\mathcal{F}$ if every connected component of $\gamma - \operatorname{sing} \mathcal{F}$ is either a leaf or transverse to $\mathcal{F}$. Furthermore, in a neighbourhood of a singularity, a transverse arc is not in a sector adjacent to an arc contained in a leaf (figure 5.5) and two transverse arcs are in two distinct sectors.

Figure 5.5:

Proposition 5.4 *There does not exist a disk D, with angular boundary, with $\partial D = \alpha \cup \beta$, where α is an arc contained in a leaf and β is a quasi-transverse arc.*

Proof. Suppose that such a disk exists. Let $N \cong D^2$, the double of D along β; N is given a foliation (with possible singularities). But now $\chi(N) > 0$, which contradicts the Euler-Poincaré formula. ∎

Remark. In the same way, we see that a closed, immersed curve quasi-transverse to $\mathcal{F}$ is not homotopic to zero. ∎

5.2 Measured foliations and simple curves

5.2.1 Notation.

$\mathcal{MF}(M)$, or simply $\mathcal{MF}$ if there is no danger of ambiguity, designates the set of measured foliations, possibly with singularities, defined on a given compact surface M, quotiented by the two following equivalence relations:

- isotopy;
- Whitehead operations

(more precisely, if the two singularities lie on the boundary, we do not contract if the connecting leaf is on the boundary.)

Recall that $\mathcal{S}$ designates the set of homotopy classes (= isotopy classes) of simple closed curves, piecewise C^∞, not homotopic to zero or to a boundary component.

5.2.2 The map $I_*\colon \mathcal{MF} \to \mathbb{R}_+^{\mathcal{S}}$.

Let $(\mathcal{F}, \mu)$ be a measured foliation and γ a closed curve. We set $\mu(\gamma) := \sup(\sum \mu(\alpha_i))$ where $\alpha_1, \ldots, \alpha_k$ are the arcs of γ, mutually disjoint and transverse to $\mathcal{F}$, and where the supremum is taken over all sums of the given type. In other words, $\mu(\gamma)$ is the total variation of the coordinate y along γ in an atlas which defines the measured foliation. This quantity is also denoted by Thurston as $\int_\gamma \mathcal{F}$.

Let σ be an element of $\mathcal{S}$; we set

$$I(\mathcal{F},\mu;\sigma) := \inf_{\gamma\in\sigma} \mu(\gamma)$$

This is clearly an isotopy invariant; on the other hand, if $(\mathcal{F},\mu)$ and $(\mathcal{F}',\mu')$ are related by a Whitehead move, then for each curve $\gamma \in \sigma$ and each $\epsilon > 0$, there exists $\gamma' \in \sigma$ such that $|\mu(\gamma) - \mu'(\gamma')| < \epsilon$ (figure 5.6).

Figure 5.6:

This is enough to ensure that the formula below defines a map:

$$\begin{aligned} &I_*\colon \mathcal{MF} \to \mathbb{R}_+^{\mathcal{S}} \\ &\langle I_*(\mathcal{F},\mu),\sigma\rangle := I(\mathcal{F},\mu;\sigma) \end{aligned}$$

5.2.3 The measure of quasi-transverse curves.

Proposition 5.5 *If γ is quasi-transverse to $\mathcal{F}$, then*

$$\mu(\gamma) = I(\mathcal{F},\mu;\sigma)$$

where σ is the homotopy class of γ.

Proof. Let $\gamma' \in \sigma$; if γ and γ' are disjoint, γ and γ' cobound an annulus A. By the Poincaré recurrence theorem (theorem 5.3), almost every leaf re-entering A in a point of γ meets the boundary again. By index considerations (proposition 5.4), it may not meet γ again. Hence $\mu(\gamma) \leq \mu(\gamma')$.

If γ and γ' have points in common, we proceed as follows. We begin by putting γ' in general position with respect to γ, in the sense that $\gamma' - \gamma$ does not contain a finite number of open intervals. This may be done in a way which perturbs the measure by an arbitrarily small amount. Then if γ and γ' are homotopic, there exists an arc α' in γ' and an arc α in γ such that $\operatorname{int}\alpha \cap \operatorname{int}\alpha' = \emptyset$ and which bound a disk D (proposition 3.10). Almost every leaf re-entrant in D in a point of α meets the boundary again. Thus $\mu(\alpha) < \mu(\alpha')$. If $\gamma' = \alpha' \cup \beta'$, we may form $\gamma'' = \alpha'' \cup \beta''$, with $\alpha'' = \alpha$ and $\beta'' = \beta'$. We have $\mu(\gamma'') \le \mu(\gamma')$ and $\pi_0(\gamma'' - \gamma) < \pi_0(\gamma' - \gamma)$. Thus, by induction on $\pi_0(\gamma' - \gamma)$, we prove that $\mu(\gamma') \ge \mu(\gamma)$. ∎

To find the classes of $\mathcal{S}$ which contain quasi-transverse curves, we require the following lemma concerning the holonomy map.

5.2.4 Stability Lemma.

Let γ be an arc (arc = compact arc) in a leaf, and let α, β be two disjoint transverse arcs, each starting from an endpoint of γ, both in the same side. Denote by L_t the leaf meeting $\alpha(t)$; $\alpha(0)$ and $\beta(0)$ are the endpoints of γ in L_0. We chose the parameterization in such a way that

$$\mu([\alpha(0), \alpha(t)]) = \mu([\beta(0), \beta(t)]) = t.$$

There exists a function germ for the holonomy map

$$h_y \colon (\alpha, \alpha(0)) \to (\beta, \beta(0))$$

characterized by the following property: h_γ is continuous and if $h_\gamma(\alpha(t))$ is defined, we have $h_\gamma(\alpha(t)) \in L_t$; the invariance of the measure μ implies that h_γ is an isometry, that is to say, that $h_\gamma(\alpha(t)) = \beta(t)$; we denote by $\{\gamma_t\}$ the continuous family of arcs, such that $\gamma_0 = \gamma$, $\gamma_t \in L_t$, and γ_t joins $\alpha(t)$ to $\beta(t)$.

Lemma 5.6 (Stability) *If h_γ is defined in the half-open interval $[\alpha(0), \alpha(t_0))$, then the points $\alpha(t_0)$ and $\beta(t_0)$ are joinable by an arc γ_{t_0} which is contained in a union of a finite number of leaves and singular points and which is the limit of the arcs $\gamma_t, t \in [0, t_0)$.*

Furthermore, there exists an immersion $H \colon [0,1] \times [0, t_0] \to M$ which is C^∞ on the interior and such that we have $H([0,1] \times \{t\}) = \gamma_t$ for all $t \in [0, t_0]$.

The obstruction to prolonging h_γ beyond $\alpha(t_0)$ is due to the following situations:

Figure 5.7:

- $\alpha(t_0)$ *(resp.* $\beta(t_0)$*) is an endpoints of* α *(resp.* β*)*
- $\gamma(t_0)$ *contains a singularity.*

Proof. We use the good atlas of 5.1.4, and the notation U_j, V_j, ϵ_0. We may clearly reduce to the case where $t_0 < \epsilon_0$, where the arc $[\alpha(0), \alpha(t_0)]$ is contained in a chart V_{j_0} and where the arc $[\beta(0), \beta(t_0)]$ is contained in a chart V_{j_1}. We then cover γ_0 by the charts $U_0 = U_{j_0}, U_1, \ldots, U_n = U_{j_1}$, the labelling is chosen in such a way as to give for each i a patch P_i^0 of U_i, contained in $U_i \cap \gamma_0$, satisfying $P_i^0 \cap P_j^0 = \emptyset$, except when $|j - i| = 1$; clearly, the labelling may go back many times to the same chart.

Consider the union $X_0 = \bigcup\{P_0^t \mid t \in [0, t_0]\}$ of patches of V_0 which cut $[\alpha(0), \alpha(t_0)]$; the eventual singularity of V_0 may not be found on the patch $P_0^{t_0}$ otherwise the holonomy map will not be defined on $[\alpha(0), \alpha(t_0))$. If we pass to the chart V_1, we find an intersection $X_0 \cap V_1$ which is a rectangle of width t_0, by the properties of a good atlas. We construct the union X_1 of patches of V_1, which meet $X_0 \cap V_1$ and we continue in this way for the rest. ∎

Remarks.

(1) The lemma requires the invariant measure; figure 5.8 is a counter-example in the case where there is non-trivial holonomy:

(2) The lemma remains true if γ_0 passes through singularities whose separatrices are on sides opposite from α and β.

Corollary 5.7 *We suppose that M is not the torus T^2. Let γ be a cycle of leaves; either γ passes through singularities and there exist separatrices on both sides of γ, or*

Figure 5.8:

γ belongs to a "maximal annulus" A whose interior leaves make cycles; a component of ∂A which is not in ∂M is a singular cycle.

Proposition 5.8 *Let γ be a simple closed (connected) curve on the surface M, and $(\mathcal{F}, \mu)$ a measured foliation. If γ separates M into two components, $M = M_1 \cup_\gamma M_2$, we denote by Σ_i $(i = 1, 2)$ a* **spine** *of M_i (i.e., a 1-complex onto which M_i collapses).* spine

1. *If $I(\mathcal{F}, \mu; [\gamma]) \neq 0$, there exists $(\mathcal{F}', \mu')$, equivalent to $(\mathcal{F}, \mu)$, such that γ is transverse to $\mathcal{F}'$ and avoids the singularities.*
2. *If $I(\mathcal{F}, \mu; [\gamma]) = 0$, there exists $(\mathcal{F}', \mu')$, equivalent to $(\mathcal{F}, \mu)$, satisfying one or the other (or both) of the following two conditions*
   - *(a) γ is a cycle of leaves of $\mathcal{F}'$;*
   - *(b) γ separates, and for $i = 1$ or $i = 2$, Σ_i is an invariant set of $\mathcal{F}'$.*

   This situation may not occur that if the set of links [???] between the leaves has cycles.

Remarks. (1) If we forbid modification of $\mathcal{F}$, we obtain only the much weaker result that γ is homotopic to an immersion quasi-transverse to $\mathcal{F}$. Moreover, this immersion is a limit of embeddings.

(2) Figure 5.9 illustrates the situation of case 2.(b) of the proposition. The foliation of the surface of genus 2 is obtained by "stretching" the curve C (see section 5.3).

We will simultaneously show the following criteria, which will be useful later on.

Minimality Criteria. The following two assertions are equivalent:

Figure 5.9:

1. $\mu(\gamma) > I(\mathcal{F}, \mu; [\gamma])$;

2. There exist two points x_0 and x_1 of γ belonging to the same leaf L in such a way that:

$$\begin{array}{rlcl} \bullet & x_0 \cup x_1 & = & \partial c, \text{where } c \text{ is an arc of } L, \\ & & = & \partial c', \text{where } c' \text{ is an arc of } \gamma \\ \bullet & c \cup c' & = & \partial D, \text{where } D \text{ is a 2-disk.} \end{array}$$

Proof. We may suppose that $\gamma = \alpha_1 * \beta_1 * \cdots * \alpha_n * \beta_n$, where the arcs α_i are transverse to $\mathcal{F}$ and the arcs β_j, eventually reduced to a point, are in a finite union of leaves and singular points; the labelling is cyclic. If we do not begin with such a decomposition, we either obtain one in each chart by an isometric isotopy, or there exists a chart in which the second conclusion of the criterion is visible and a correction shortening the length leads to a finite decomposition.

This done, we have for β_k the configuration shown in figure 5.10. By definition, (1) is the configuration for which lemma 5.6 applies. In (2) and (3), β_k contains at least one singularity, and lemma 5.6 is not applicable; in (4), β_k does not contain any singularities.

In (1), the conclusion of the second criterion is visible. On the other hand, we claim that, *if for all k, β_k is not in configuration (1), then γ is isotopic to a quasi-transverse curve of the same length*; that is to say, $\mu(\gamma)$ is minimal by 5.5, and the claim proves the minimality criterion. To prove the claim, we replace each configuration of type (4) by a transverse arc; the configurations (2) and (3) are modified as in figure 5.11.

∎

Figure 5.10:

Figure 5.11:

We will now give the proof of the proposition. As in the proof above, the β_k of type (4) are replaced by transverse arcs, and those of types (2) and (3) may be supposed to have singularities as endpoints.

At this point, either γ is a cycle of leaves (conclusion 2.(a) of the proposition), or we may concentrate in a point each leaf contained in γ. By then resolving [???] the singularities obtained (figure 5.12), we reduce to the situation where *all of the arcs β_k are of type (1).* From this, the induction is made on the number of arcs of γ contained in a leaf. When there are none, γ is transverse to the foliation (conclusion 1.)

If not, consider β_1. Lemma 5.6 gives an immersion h of a rectangle R. The induced foliation $\hat{\mathcal{F}} = h^{-1}(\gamma)$ has all of its singularities in the same arc λ of the boundary. Denote by $\hat{\beta}_1, \ldots, \hat{\beta}_m$ the arcs of $\hat{\gamma} = h^{-1}(\gamma)$, which are in the leaves of $\hat{\mathcal{F}}$ (horizontal arcs). Say that $\hat{\beta}_1$ is the closest arc to the singularities (in the

Figure 5.12:

sense of the transverse measure); then the component of $\hat{\gamma}$ which contains $\hat{\beta}_1$, bounds a sub-rectangle R' which is minimal; we see that $h|\operatorname{int}(R')$ is an embedding disjoint from γ.

Figure 5.13:

If R' does not contain any singularities of $\hat{\mathcal{F}}$, a neighbourhood of $h(R')$ is the support for an isotopy of γ which pushes out $\hat{\beta}_1$; also if $h(\hat{\beta}_1) = \beta_1$, the application of this isotopy leads to a situation where, in the new rectangle R associated with β_1, the new arc $\hat{\gamma}$ has fewer horizontal arcs.

If R has one singularity, then, because of the transverse measure, it is easy to see that $h(\hat{\beta}_1)$ is an arc β_k distinct from β_1 (if not, the width of R' becomes the same as that of R).

By the reasoning above, perhaps after cyclically re-labelling the arcs, we may suppose that:

1. $h|R - \lambda$ is an embedding,
2. $h|\operatorname{int}(R) \cap \gamma$ is empty,
3. $h(\lambda) \cap \beta_k$ is empty, for all k.

Inserting first the following simple cases (A) and (B), where we see the operations of isotopy and Whitehead moves, which reduce the number of arcs of γ contained in a leaf.

(A) λ does not contain a singularity.

See figure 5.14.

Figure 5.14:

(B) λ contains singularities and R is embedded.

The isotopy across R replaces β_1 with an arc of type (2). We apply to it the procedure from the beginning [???].

This case makes allowances [??? mis à part...] that $h(\lambda)$ has double points. Seen as a singular path, λ is written as a composite:

$$\lambda = \mu_0 * \lambda_1 * \cdots * \lambda_q * \mu_1$$

where μ_0 (resp. μ_1) is an arc of a leaf joining a point of α_1 (resp. α_2) to a singularitym and where λ_i ($1 \leq i \leq q$) is an arc of a leaf joining two singularities; certain of these arcs may be reduced to a point and several may belong to the same leaf. However, λ has in R an approximation which is an embedded arc missing α_1 and α_2 at its endpoints. Because of this, each leaf carries at least two arcs of λ. More precisely, neither μ_0 nor μ_1 may belong to the same leaf [??? as] λ_j; if $\mu_0 \cap \mu_1$ is not reduced to one of their endpoints, it is that $\alpha_1 = \alpha_2$ (i.e., $\gamma = \alpha_1 * \beta_1$) and that we have the configuration of figure 5.15.

We say that λ_j is **simple** if, for every $j' \neq j$, λ_j does not cover the same leaf as λ'_j. We say that μ_0 and μ_1 are simple if one does not have the configuration of figure 5.15 ([???] it is then compelled on the other). simple

Figure 5.15:

Denote by Λ the complex of dimension 1

$$\Lambda = \bigcup_{i=0}^{q} \lambda_i$$

this is an invariant set of the foliation $\mathcal{F}$. If M is closed, each Whitehead slide of Λ lifts to Whitehead moves of F (the terminology for foliations was chosen owing to this remark.)

Lemma 5.9 *We suppose that M is closed. If one of the arcs λ_i, μ_0, μ_1 is simple, there exists a foliation $\mathcal{F}'$ equivalent to $\mathcal{F}$ and equal to $\mathcal{F}$ in the complement of a neighbourhood of Λ, and for which the limiting arc λ' of the domain of deformation of R' of β_1 possess fewer double simplexes (edge or vertex).*

Proof. We slide the simple arc on its predecessor or on its successor. Figure 5.16 represents this operation when μ_0 is simple. ∎

If the lemma is applicable, we reduce by iteration to case (B), if not, we find ourselves in the following situation.

Figure 5.16:

(C) All of the arcs λ_i, μ_0, μ_1 are double.

Then the support of R, on the surface, is a regular neighbourhood of the complex Λ and γ is its boundary; we thus have the conclusion 2.(b) of the proposition.

The proof of the proposition is completed by induction on the number of segments of the decomposition of γ, all as long as M is closed. The case for surfaces with boundary is made analogously, paying attention to the Whitehead slides permitted.

∎

Remark. The preceding proposition does not admit the reasonable generalization to the case of a system with k embedded curves $\gamma_1, \ldots, \gamma_k$; except when $I(\mathcal{F}, \mu; [\gamma_1]) \neq 0, \ldots, I(\mathcal{F}, \mu; [\gamma_{k-1}]) \neq 0, I(\mathcal{F}, \mu; [\gamma_k])$ maybe null.

5.3 Curves as Measured foliations

5.3.1 The "stretching" procedure.

Let M_0 be a sub-manifold of dimension 2 in M, such that $M - M_0$ does not have any contractible components. Let Σ be a **spine** of $\overline{M - M_0}$; by hypothesis, none of its components are contractible. Thus, perhaps after collapsing the 1-simplices which have a free vertex, each singularity of Σ involves three branches. spine

We may construct a surjective map $j\colon M_0 \to M$ such that

- j is a (piecewise differentiable) immersion.

- $j|\operatorname{int}(M_0)$ is a diffeomorphism on $M - \Sigma$.
- $j(\partial M_0 - \partial M) = \Sigma$.
- j is the identity outside of a small collar around $\partial M_0 - \partial M$.

Figure 5.17:

Let $\mathcal{F}_0$ be a measured foliation on M_0 such that each component of $\partial M_0 - \partial M$ is an invariant set. We may then define $\mathcal{F} := j_*\mathcal{F}_0$ which is a measured foliation on M satisfying

- Σ is an invariant set of $\mathcal{F}$.
- $j|\operatorname{int}(M_0)$ conjugates between the measured foliations $\mathcal{F}_0|\operatorname{int}(M_0)$ and $\mathcal{F}|(M - \Sigma)$. We say that $\mathcal{F}$ is obtained from $\mathcal{F}_0$ by stretching M_0.

We observe that if Σ' is another spine of $\overline{M - M_0}$, then Σ' is obtained from Σ by Whitehead operations and isotopies (see appendix B). We conclude that the class of $\mathcal{F}$ does not depend on that of $\mathcal{F}_0$. We therefore may define a map

$$\mathcal{MF}(M_0, \partial M_0 = \partial M) \to \mathcal{MF}(M)$$

for which the domain is the subset of $\mathcal{MF}(M_0)$ formed from the foliations admitting every component of $\partial M_0 - \partial M$ as an invariant set.

Lemma 5.10 *Let μ_0 and μ_1 be transverse invariant measures for $\mathcal{F}_0$ and $\mathcal{F}$. Let γ be a simple curve in M. Then $I(\mathcal{F}, \mu; [\gamma]) = \inf \mu_0(\gamma' \cap M_0)$, where γ' is isotopic to γ.*

Proof. This follows from the observation that for each curve C, there exists a curve C', isotopic to C, such that $C' \cap M_0 = j^{-1}(C)$. ∎

5.3.2 The inclusion $\mathbb{R}_+ \times \mathcal{S} \hookrightarrow \mathcal{MF}$.

We will define the inclusion $\mathbb{R}_+ \times \mathcal{S} \hookrightarrow \mathcal{MF}$ as follows. Let $C \in \mathcal{S}$, $\lambda \in \mathbb{R}_+^*$. Consider a tubular neighbourhood M_0 of C which we foliate by circles parallel to C; we equip this with an invariant transverse measure μ_0 such that the width of the annulus M_0 is λ. This measured foliation of M_0 is unique up to isotopy. We denote by $F_{\lambda,C}$ a foliation obtained in this way, and by μ its transverse measure.

Proposition 5.11 *Let γ be a simple curve in M. Then we have*

$$I(\mathcal{F}_{\lambda,C}, \mu; [\gamma]) = \lambda i(C, \gamma)$$

Proof. Let α be a component of $\gamma \cap M_0$. If α goes to one boundary or the other of M_0, then $\mu_0(\alpha) \geq \lambda$. We deform α by isotopy so as to be transverse to the foliation; then $\alpha \cap C = 1$ point and $\mu_0(\alpha) = \lambda$. If α does not touch a component of the boundary, then γ is isotopic to γ' whose intersection with M_0 has one less component. Applying the preceding lemma, we have the inequality

$$I(\mathcal{F}_{\lambda,C}, \mu; [\gamma]) \geq \lambda i(C, \gamma).$$

The equality is obtained by considering the case where γ has minimal intersection with C, for then

$$\mu_0(\gamma) = \lambda i(C, \gamma).$$

∎

The proposition above indicates that the following diagram commutes:

$$\mathbb{R}_+^* \times \mathcal{S} \to \mathcal{MF}$$

As i_* is injective (by proposition 3.18), $\mathbb{R}_+^* \times \mathcal{S} \to \mathcal{MF}$ is also an injection.

Chapter 6

Measured Foliations, Continued

by A. Fathi

Chapter 7
Teichmüller Space

by A. Douady

Notes by F. Laudenbach

Given a compact surface M of Euler characteristic $\chi(M) < 0$, we consider the space $\mathcal{H}$ of metrics of curvature -1 on M, which make the boundary of M geodesic; $\mathcal{H}$ is non-empty, and is given the C^∞ topology of all fields of covariant tensors. The group $\mathrm{Diff}^0(M)$ of diffeomorphisms of M isotopic to the identity, equipped with the C^∞ topology, acts on the left on $\mathcal{H}$ by the general formula obtained from the naturality of the field of covariant tensors $m \in \mathcal{H}$, $\phi \in \mathrm{Diff}^0(M) \to \phi^* m \in \mathcal{H}$. The quotient space $\mathcal{T} := \mathcal{H}/\mathrm{Diff}^0(M)$ is the **Teichmüller space** of M; when M is orientable, this definition coincides with the classical definition as the "space of complex structures, up to isotopy", via the uniformization theorem [Spr66]. It is known that this space is homeomorphic to a "cell" [FK98]; Earle and Eells have shown that $\mathcal{H}$ is the total space of a principal fibration over the Teichmüller space [EE69]. Teich müller space

Our programme here is to establish a parameterization of the Teichmüller space which depends only on the lengths of the simple closed geodesics.

Recall that $\mathcal{S}$ is the set of isotopy classes of simple closed curves not homotopic to zero in M. If m is a hyperbolic metric, for $\alpha \in \mathcal{S}$, $\ell(m, \alpha)$ is the length of the unique geodesic in the isotopy class of α. This defines the map

$$\ell_* : \mathcal{T} \to \mathbb{R}_+^{\mathcal{S}}$$

by the formula $\langle \ell_*(m), \alpha \rangle := \ell(m, \alpha)$.

Proposition 7.1 *For a fixed $\alpha \in \mathcal{S}$, the map which associates to $m \in \mathcal{H}$ the m-geodesic in the class α is continuous in the C^∞ topology.*

Corollary 7.2 *The map ℓ_* is continuous.*

Proof. Clear ∎

Proof. [of proposition 7.1.] One method of proof employs convexity properties of the "displacement function" (the Bishop-O'Neill Theorem [BO69]; see the paper by Bourguignon in [Bou85]). We give a different proof.

Let Γ be the set of pairs (m, γ) where m is a hyperbolic metric and $\gamma\colon S^1 \to M$ is a constant speed parameterization of the m-geodesic of α. We give Γ the topology induced from the C^∞ topology on the product space

$$\mathcal{H} \times C^\infty(S^1, M).$$

Consider the projection $p\colon \Gamma \to \mathcal{H}$ onto the first factor. We wish to show that p is proper.

We denote by TM the manifold of tangent vectors to M, and consider in $\mathcal{H} \times TM$ the set

$$\begin{aligned} C \quad := \quad & \{(m, v) \mid \forall t, \exp_m (t+1)v = \exp_m tv \\ & \qquad \text{and the closed curve } t \in [0,1] \to \exp_m tv \text{ is in the class } \alpha\} \end{aligned}$$

C is closed in the product topology on $\mathcal{H} \times TM$. If S^1 is obtained by identifying the endpoints of $[0, 1]$, one has the obvious map $C \to \Gamma$ which is surjective; by a theorem on differential equations, it is continuous. The properness of p follows from the fact that the projection $q\colon C \to \mathcal{H}$ is proper, as we shall prove.

We know that $m \in \mathcal{H} \mapsto \ell(m, \alpha)$ is an upper semi-continuous function. Hence if m belongs to a compact set K, the set $\{\ell(m, \alpha) \mid m \in K\}$ is bounded. Let $(m, v) \in q^{-1}(K)$; the quantity $\sqrt{m(v, v)} = \ell(m, \alpha)$ is then bounded. If $m_0 \in K$, there exists $\lambda > 0$ such that, for all $w \in TM$, and all $m \in K$, one has

$$m_0(w, w) \leq \lambda m(w, w).$$

Thus if $(m, v) \in q^{-1}(K)$, $m_0(v, v)$ is bounded. Finally, $q^{-1}(K)$ is compact, since it is closed in a product of compact sets.

The group $O(2)$ of rotations acts naturally on Γ: for $r \in O(2)$, $(m, \gamma) * r = (m, \gamma \circ r)$. The quotient is the space of m-geodesics of α, $m \in \mathcal{H}$. In negative curvature, p induces a *bijection* $\Gamma / O(2) \to \mathcal{H}$, which, by the above, is continuous and proper. Since the spaces considered are metrizable, the inverse is also continuous. ∎

From now on, we suppose that M is without boundary, in order to simplify our presentation. Let g be the genus of M. We fix a decomposition $\mathcal{K}$ of M into pairs of pants R_i, $i = 1, \ldots, 2g-2$, bounded by curves $K_j, j = 1, \ldots, 3g-3$. Every pair of pants is given with a parameterization by a modulus, and every curve K_j is given with an orientation. We have a continuous map

$$L \colon \mathcal{T} \to (\mathbb{R}_+^*)^{3g-3}$$

defined by $L(m) := \ell(m, K_i); (i = 1, \ldots, 3g-3)$, where m is a hyperbolic metric making the K_i's geodesic (a so-called **metric adapted to the decomposition**.)

metric adapted to the decomposition

Remark. From now on, $\mathcal{H}$ denotes the space of metrics adapted to $\mathcal{K}$. One sees easily that $\mathcal{T}$ is bijective with the quotient of $\mathcal{H}$ by $\mathrm{Diff}(M, \mathcal{K}) \cap \mathrm{Diff}^0(M)$. To see that the topology is the same, we use proposition 1 and the fact that the action of $\mathrm{Diff}(M)$ on the space of simple curves admits local sections [Pal60].

The **twist** along the curves K_i defines a continuous action θ of $\mathbb{R}^{3g-3}$ on $\mathcal{T}$. More precisely, let $K_i \times [0,1]$ be a collar for $K_i = K_i \times \{0\}$, given once and for all; we suppose all of the collars are pairwise disjoint. Given an adapted hyperbolic metric m and a number α, there is a diffeomorphism $\phi_i(m, \alpha)$ of the collar $K_i \times [0,1]$ with the following properties:

twist

1. $\phi_i(m, \alpha)$ is the identity on a neighbourhood of $K_i \times \{1\}$;
2. $\phi_i(m, \alpha)$ is an isometry for m in a neighbourhood of $K_i \times \{0\}$;
3. The lift of $\phi_i(m, \alpha)$ to the universal covering $\mathbb{R} \times [0,1]$ which is the identity on $\mathbb{R} \times \{1\}$ is a translation of length $\alpha \ell(m, K_i)$ on $\mathbb{R} \times \{0\}$, in the sense indicated by the sign of α (the universal covering is given the lifted metric).

The twisted metric $\phi_i(m,\alpha)$ is defined by $\theta_i(m,\alpha) = \theta_i^*(m,\alpha)m$ for points of the collar $K_i \times [0,1]$ and by $\phi_i(m,\alpha)$ everywhere else.

For $(\alpha_1,\ldots,\alpha_{3g-3}) \in \mathbb{R}^{3g-3}$, let $\theta(m,\alpha_1,\ldots,\alpha_{3g-3})$ be the metric defined by $\theta_1(m,\alpha_1)$ in $K_1 \times [0,1],\ldots$, by $\theta_{3g-3}(m,\alpha_{3g-3})$ in $K_{3g-3} \times [0,1]$, and m elsewhere. Because of the adapted metric, its isotopy class is well defined.

Remarks. (1) As in the classification of metrics on pairs of pants (exposé 3), the orbits of the action θ coincide exactly with the fibres of L. The corollary of proposition 2 implies that this action is free.

Dehn twist

(2) The **Dehn twist** along K_i, which is a global diffeomorphism of the surface supported in a collar of K_i, is an isometry (up to isotopy) for the metric $\theta_i(m,1)$ on m. One therefore has, for all curves K',

$$\ell(\theta_i(m,1),[K']) = \ell(m,\rho([K'])).$$

Let R and R' be the two pairs of pants adjacent to K_i; suppose that R contains the collar $K_i \times [0,1]$. Let K_i' be a simple curve in $R \cup R'$ cutting K_i in two essential points (K_i' is not isotopic to a curve disjoint from K_i.) — compare with section 4.4 of exposé 6. Let K_i'' denote the curve in $R \cup R'$ obtained from K_i' by a Dehn twist along K_i: $K_i'' := \rho(K_i')$.

Proposition 7.3 *The length $\ell(\theta_i(m,\alpha),[K_i'])$ is a strictly convex function of α, which takes a minimum.*

Corollary 7.4 *(1.) Given the metric m_0, there exists an isotopy class γ_i in $R \cup R'$ such that the function*

$$\alpha \mapsto \ell(\theta_i(m_0,\alpha),\gamma_i)$$

is strictly increasing for $\alpha > 0$.

(2.) The length $\ell(\theta_i(m,\alpha),[K_i'])$ tends uniformly to $+\infty$ as α tends to $+\infty$ or $-\infty$ and m remains in a compact set.

Proof of corollary. (1.) Suppose that $\ell(\theta_i(m,\alpha),[K_i'])$ is increasing from $\alpha = k$, where k is an integer. We then take $\gamma_i = \rho^k([K_i'])$ and apply remark (2) above.

(2.) This is a general property of families of functions of a real variable which are strictly convex and take a minimum, and which depend continuously on a parameter (in the compact open topology). Let $f_\lambda(x)$ be such a

Figure 7.1: Proof of lemma 7.5

family, and let $x = m(\lambda)$ be the point where the functions realize their minimum. Then $m(\lambda)$ is a continuous function. In effect, given ϵ, if λ is sufficiently near to λ_0, one has

$$f_\lambda(m(\lambda_0)) < \inf[f_\lambda(m_{\lambda_0} - \epsilon), f_\lambda(m_{\lambda_0} + \epsilon)];$$

thus $m(\lambda)$ belongs to the open interval $]m(\lambda_0) - \epsilon, m(\lambda_0) + \epsilon[$. Now let $x_0 > m(\lambda_0)$ and let K lie between $f_{\lambda_0}(m(\lambda_0))$ and $f_{\lambda_0}(x_0)$. Then if λ is sufficiently close to λ_0, one has $f_\lambda(x_0) > k$ and f_λ is strictly increasing on $[x_0, +\infty]$; thus $f_\lambda([x_0, +\infty)) \subset]k, +\infty)$. ∎

Proof of proposition.

Lemma 7.5 *Let γ be a line in the Poincaré half-plane, and let τ be an isometry leaving γ invariant. Let x be a point of γ and y a point not on γ. Then*

$$d(x, \tau x) < d(y, \tau y),$$

where d denotes hyperbolic distance.

Proof. One may take for x the base of the perpendicular α to γ passing through y. Then γ is the unique common perpendicular of α and $\tau\alpha$. This gives the inequality. ∎

Lemma 7.6 *Let γ_1 and γ_2 be two lines which do not intersect in hyperbolic space. The function $d(x, y)$, $x \in \gamma_1$, $y \in \gamma_2$, is strictly convex.*

Proof. Let x, x' (resp. y, y') be two points of γ_1 (resp. γ_2); without loss of generality, suppose that $x \neq x'$. Let i be the middle of the arc $\overline{xx'}$, j the middle of $\overline{yy'}$, δ the geodesic $\overline{ij}$. Denote by σ_i (resp. σ_j) the symmetry with respect to i (respectively j); $\sigma_j\sigma_i$ is an isometry which leaves δ invariant. Let $z = \sigma_j\sigma_i(x)$, $z' = \sigma_j\sigma_i(x')$, $k = \sigma_j\sigma_i(i)$. Then σ_j takes x to z' and y to y'. Therefore

$$d(x, y) = d(y', z').$$

The triangle inequality gives

$$d(x', z') \leq d(x', y') + d(y', z').$$

By lemma 1, one has

$$2d(i, j) = d(i, k) < d(x', z').$$

(Note that γ_1 does not cut γ_2, hence the point x' is not on δ). Finally, we obtain the inequality giving convexity:

$$2d(i, j) < d(x, y) + d(x', y').$$

∎

Conclusion of proof of prop. 2

Since the surface given is equipped with the metric m and its universal covering is identified with the hyperbolic space $\mathbb{H}^2$, there exists an element τ of $\pi_1(M, *)$ which acts as an isometry on $\mathbb{H}^2$, leaving invariant a line δ, which is the lift of the geodesic K_i'. Let x be a point of δ projecting to a point of $K_i' \cap K_i$; denote by $\widetilde{K}^1$ the lift of K_i by x and let $\widetilde{K}^3$ denote the lift by τx. The segment $(x, \tau x)$ cuts exactly another lift $\widetilde{K}^2$ of K_i in a point y. In figure 7.1, we show the orientations which are given the three lifts.

If one twists the metric by an angle α in the collars indicated in figure 7.2, the lift of the $\theta_i(m, \alpha)$-geodesic of $[K_i']$ cuts $\widetilde{K}^1$ in a point x' and $\widetilde{K}^2$ in y'; it is a line from x' to y' with the metric of the hyperbolic plane; but from y' to $\tau x'$ its length is the hyperbolic distance $d(y' + \alpha, \tau x' + \alpha)$; in this formula, "+" designates the translation along the geodesics $\widetilde{K}^2$ and $\widetilde{K}^3$. Finally, one has

$$\ell(\theta_i(m, \alpha), [K_i']) = \inf_{x \in \widetilde{K}^1, y \in \widetilde{K}^2} (d(x, y) + d(y + \alpha, \tau x + \alpha)).$$

Figure 7.2:

We now show that $f(x, y, \alpha) := d(x, y) + d(y + \alpha, \tau x + \alpha)$ is a proper, strictly convex function. For this, we use the fact that $d(x, y)$ is proper, because the lines on which the points are moved have a common perpendicular (at a finite distance), and by lemma 2 it is strictly convex.

To show that f is proper, let $(x_n, y_n, \alpha_n) \to \infty$; if $(x_n, y_n) \to \infty$ then $d(x_n, y_n) \to +\infty$; hence $f(x_n, y_n, \alpha_n) \to +\infty$. If (x_n, y_n) remains in a compact set, then $\alpha_n \to \infty$ and $(y_n + \alpha_n, \tau x_n + \alpha_n)$ tends to ∞, hence $d(y_n + \alpha_n, \tau x_n + \alpha_n)$ tends to $+\infty$.

One verifies immediately that f is strictly convex.

For α fixed, the function $f(x, y, \alpha)$ has a minimum $g(\alpha)$ since f is proper. The convexity of f implies that g is also convex; since $g(\alpha)$ is a value attained by $f(x, y, \alpha)$, one verifies that g is strictly convex.

The function f has an absolute minimum (f is proper and bounded below; it is the minimum of g.) ∎

Proposition 7.7 *The map $L\colon \mathcal{T} \to (\mathbb{R}_+^*)^{3g-3}$ is a principle fibration of the group $\mathbb{R}^{3g-3}$ acting by θ.*

Corollary 7.8 *The Teichmüller space of a closed surface of genus g is homeomorphic to $\mathbb{R}^{6g-6}$.*

Proof. The important point is to show that there exist local sections for L. We know from theorem 5 in exposé 3 that, for the pair of pants P^2, the map

$$\mathcal{H}(P^2) \to (R_+^*)^3$$

(which associates to a metric adapted to the boundary the lengths of the three boundary components) admits local sections at the level of the metrics.

We know that to glue together two hyperbolic metrics along a geodesic, it is enough to specify an isometry of the geodesic along which we glue. Now if one has a metric on P^2 and if one considers a curve C on the boundary, one has a unique geodesic arc without a double point which meets C in its two endpoints (an arc *jaune*[1]). By proposition 7.1, its origin (which one distinguishes from the other endpoint by an orientation chosen once and for all) varies continuously with the metric.

The sought-for local section may now be obtained. Above the $3g-3$ lengths one chooses a metric with the following property: if K_j is adjacent to R_{i_1} and R_{i_2} the two origins on K_j of the *arcs jaune* of the two pairs of pants coincide. In imposing this condition, we obtain a continuous local section.

Let D be a ball of $(R_+^*)^{3g-3}$ over which L admits a section σ. Define a map $T\colon D \times \mathbb{R}^{3g-3} \to \mathcal{T}$ by

$$T(x, \alpha_1, \ldots, \alpha_{3g-3}) := \theta(\sigma(x), \alpha_1, \ldots, \alpha_{3g-3})$$

It remains to show that T is a homeomorphism onto its image. Since $\mathcal{T}$ has a countable neighbourhood base, it is enough to show that T is injective and proper.

If the two metrics differ only by a twist, they are distinguished by a length of a geodesic (corollary to proposition 2); this proves injectivity.

To simplify notation, let $(\alpha_1, \ldots, \alpha_{3g-3})$ be written as α. Let (x^n, α^n) be a sequence tending to infinity in $D \times \mathbb{R}^{3g-3}$. The second part of the corollary just cited forbids that the image under T of this sequence be a compact set in Teichmüller space. Hence T is proper. ∎

Theorem 7.9 *The map* $\ell_*\colon \mathcal{T} \to \mathbb{R}_+^{\mathcal{S}}$ *is a proper map which is a homeomorphism onto its image.*

We will actually prove a much stronger proposition, in terms of the system of curves K_i, K_i', K_i'' introduced in the preamble to proposition 2.

[1] Compare with the terminology of measured foliations (exposé 6).

Proposition 7.10 *The map* $\Lambda\colon \mathcal{T} \to \mathbb{R}_+^{9g-9}$ *which associates to the point* $m \in \mathcal{T}$ *the triple* $(\ell(m,[K_i]),\ \ell(m,[K_i']),\ \ell(m,[K_i'']))$ *is injective and proper (hence a homeomorphism onto its image).*

Proof. We choose a section s of the fibration L; that is, we write every $m \in \mathcal{T}$ in the form

$$m = \theta(s(x), \alpha)$$

where $\alpha = (\alpha_1, \ldots, \alpha_{3g-3}) \in \mathbb{R}^{3g-3}$ is a "multi-angle" of twist, and where $x \in (\mathbb{R}_+^*)^{3g-3}$ is the system of lengths of the curves K_i.

The variable x being fixed, the function $\ell(m,[K_i'])$ is a strictly convex, proper function $g_i(\alpha_i)$ of the i-th component of α; furthermore, $\ell(m,[K_i'']) = g_i(\alpha_i + 1)$.

Lemma 7.11 *If* $g\colon \mathbb{R} \to \mathbb{R}$ *is a strictly convex proper function then* $t \mapsto (g(t), g(t+1))$ *defines a proper immersion of* $\mathbb{R}$ *into* $\mathbb{R}^2$.

Thus the $(6g-6)$*-system* $(\ell(m,[K_i']), \ell(m,[K_i'']))$ *is an injective proper function of the multi-angle* α*. From this, it follows that* Λ *is injective.*

To show that Λ is proper, we consider the sequence (x_n, α_n) tending to ∞. If x_n tends to ∞, it is clear that $\Lambda(x_n, \alpha_n)$ tends to ∞; otherwise x_n remains in a compact set and by corollary 7.4, the length of one of the curves K_i' tends to infinity. ∎

We complete the theorem by the following proposition, following a proof indicated by S. Kerckhoff. Recall that π denotes the projection $\mathbb{R}^{\mathcal{S}} \setminus \{0\} \to P(\mathbb{R}^{\mathcal{S}})$.

Proposition 7.12 *The composite map* $\pi \circ \ell_*\colon \mathcal{T} \to P(\mathbb{R}_+^{\mathcal{S}})$ *is an injection.*

Proof. We take for $\mathbb{H}^2$ the half-plane model of Poincaré: $\{x + iy \mid y > 0\}$, with the metric $ds := \frac{dx^2+dy^2}{y^2}$. The group of isometries is $SL(2,\mathbb{R})/\{\pm \mathrm{Id}\}$ where the action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $z \mapsto \frac{az+b}{cz+d}$.

If A is a hyperbolic element (i.e., it leaves invariant a geodesic), we define the **displacement** displacement

$$\ell(A) := \inf_{z \in \mathbb{H}^2} d(z, A \cdot z)$$

The minimum is attained on the invariant geodesic. ∎

Lemma 7.13 *If $A \in SL(2,\mathbb{R})$ is hyperbolic, one has:*

$$\mathrm{Tr}(A) = 2\cosh(\frac{\ell(A)}{2})$$

∎

Proof. By conjugating within $SL(2,\mathbb{R})$ we reduce to the case where the invariant geodesic is the y half-axis. One then has

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \qquad \rho > 0$$

Consequently, $A \cdot i = \rho^2 i$. Thus, we have

$$\ell(A) = d(i, \rho^2 i) = \int_1^{\rho^2} \frac{dt}{t} = 2\log(\rho)$$

and

$$\mathrm{Tr}(A) = \rho + \rho^{-1} = 2\cosh(\frac{\ell(A)}{2})$$

∎

Lemma 7.14 *Let $A, B \in SL(2,\mathbb{R})$. One has*

$$\mathrm{Tr}(A) \cdot \mathrm{Tr}(B) = \mathrm{Tr}(AB) + \mathrm{Tr}(A^{-1}B)$$

This follows from a direct calculation. ∎

Consider on the surface M two simple oriented curves γ_1 and γ_2 which intersect transversely in a base-point.

We may hence speak of the homotopy classes of the pointed loops $\gamma_1 * \gamma_2$ and $\gamma_1^{-1} * \gamma_2$; both are representable by simple curves γ_3 and γ_4. If M is given a metric m of curvature -1, these elements of the fundamental group correspond to hyperbolic isometries of $\mathbb{H}^2$ for which the displacement is $\ell_i := \ell(m, [\gamma_i])$. The preceding lemmas thus give the formulas

$$2\cosh(\frac{\ell_1}{2})\cosh(\frac{\ell_2}{2}) = \cosh(\frac{\ell_3}{2})\cosh(\frac{\ell_4}{2})$$

and

$$\cosh(\frac{\ell_1+\ell_2}{2}) + \cosh(\frac{\ell_1-\ell_2}{2}) = \cosh(\frac{\ell_3}{2}) + \cosh(\frac{\ell_4}{2}) \tag{7.1}$$

(**H**) Suppose that there is another metric of curvature -1 for which the lengths of all closed geodesics are multiplied by $k \neq 1$. For this metric, the equality **??** changes to

$$\cosh(k\frac{\ell_1+\ell_2}{2}) + \cosh(k\frac{\ell_1-\ell_2}{2}) = \cosh(k\frac{\ell_3}{2}) + \cosh(k\frac{\ell_4}{2}) \tag{7.2}$$

Lemma 7.15 *Let $\alpha, \beta, \gamma, \delta$ be four non-negative numbers and let $k > 0, k \neq 1$. The relations*

$$\begin{aligned} \cosh\alpha + \cosh\beta &= \cosh\gamma + \cosh\delta \\ \cosh k\alpha + \cosh k\beta &= \cosh k\gamma + \cosh k\delta \end{aligned}$$

imply that $\{\alpha, \beta\} = \{\gamma, \delta\}$.

Proof. One may restrict to $k > 1$. The reader may check that the function $\cosh(k\mathrm{Arg}\cosh \mathrm{x})$ is a strictly convex function of x. Then, if c is a common value of the first inequality and if one sets $x = \cosh\alpha, y = \cosh\gamma$, the second relation is

$$\begin{aligned} &\cosh(k\mathrm{Arg}\cosh \mathrm{x}) + \cosh(\mathrm{kArg}\cosh(\mathrm{c}-\mathrm{x})) \\ &= \cosh(k\mathrm{Arg}\cosh \mathrm{y}) + \cosh(\mathrm{kArg}\cosh(\mathrm{c}-\mathrm{y})) \end{aligned}$$

One may suppose that $y \leq x \leq c - x \leq c - y$. If $y < x$, by strict convexity, the left side is strictly less than the right. ▮

Consequently, (**??**) and (**??**) give

$$\{\ell_1+\ell_2, \ell_1-\ell_2\} = \{\ell_3, \ell_4\}$$

By a change of notation, we may write

$$\ell_3 = \ell_1 + \ell_2$$

Since the angle between γ_1 and γ_2 is nonzero, it is not possible for $\ell_1 + \ell_2$ to be a shorter distance; hence, the above inequality can't be true, and the hypothesis (**H**) is absurd.

Chapter 8

Thurston's Compactification of Teichmüller Space

by A. Fathi and F. Laudenbach

Chapter 9

Classification of Surface Diffeomorphisms

by V. Poénaru

9.1 Preliminaries

Let M be a closed, orientable surface of genus $g > 1$. Its compactified Teichmüller space $\overline{\mathcal{T}M}$ is homeomorphic to D^{6g-6}. The natural action of $\pi_0(\mathrm{Diff}(M))$ on M and $\mathcal{PMT}(M)$ combine to give a *continuous* action on

$$\overline{\mathcal{T}M} = \mathcal{T}M \cup \mathcal{PMF}(M)$$

Let $\varphi \in$ Diff(M) and let $[\varphi]$ be its isotopy class. By Brouwer's fixed point theorem, there is an $x \in \overline{\mathcal{T}M}$ such that $[\varphi] \cdot x = x$.

If x is an element of $\mathcal{T}M$, then x gives a hyperbolic metric on M, well-defined up to isotopy, and φ is isotopic to an isometry in this metric. By theorem ?? in exposé 3, φ is isotopic to a diffeomorphism of finite order.

If x belongs to the boundary of M, $x \in \mathcal{PMF}(M)$; the equality $[\varphi] \cdot x = x$ tells us that there exists a measured foliation whose measure class in the projective space $P(\mathbb{R}_+^{\mathcal{S}})$ is preserved by φ. [Notation: $\mathcal{F}$ denotes the foliation and μ denotes a transverse invariant measure on F.] In other words, there exists a measured foliation $(\mathcal{F}, \mu)$ and a scalar $\lambda \in \mathbb{R}_+$ such that

$$\varphi(\mathcal{F}, \mu) \sim_m \varphi(\mathcal{F}, \lambda\mu) = \lambda(\mathcal{F}, \mu) \qquad (9.1)$$

Notes: (1) Here $\sim_m$ is the relation 'equivalence in measure' between measured foliations. Recall that

$$(\mathcal{F}_1, \mu_1) \sim_m (\mathcal{F}_2, \mu_2)$$

tells us that the two measured foliations define the same functional in $\mathbb{R}_+^{\mathcal{S}}$ (equivalence in the sense of Schwartz). By the results of exposé 6, this relation is the same as equivalence in the sense of Whitehead, defined in II.1 in exposé 5.

(2) $\varphi(\mathcal{F}, \mu)$ denotes the image foliation of $\mathcal{F}$ under φ, equipped with the [image-direct] measure: the measure of a transverse arc α is the μ-measure of $\varphi^{-1}(\alpha)$.

To go any further, I define a partial measured foliation of M, which is given on a compact submanifold N of dimension 2, and by a measured foliation $(\mathcal{F}', \mu')$ supported on N, satisfying the following:

(i) Every connected component of ∂N is a cycle of leaves.

(ii) If Γ is a component of ∂N which bounds a disk in $M \setminus \operatorname{int}(N)$, then the number of separatrices which belong to the set $\operatorname{Sing}(\mathcal{F}' \cap \Gamma)$, re-entrant in N, is at least 2.

Given part of a measured foliation $(\mathcal{F}', \mu)$ of M, we may "unglue" $\mathcal{F}$ along all of the leaves which join the singularities, and "blow-up" the singularities which are not touched by the connection. We obtain, then, a partial measured foliation $U(\mathcal{F}, \mu)$, called the **unglue** of $(\mathcal{F}, \mu)$, whose singularities are all on the boundary. One easily verifies the following facts: unglue

(a) $i_*(\mathcal{F}, \mu) = i_*(U(\mathcal{F}, \mu)) \in R_+^{\mathcal{S}}$

(b) if $i_*(\mathcal{F}_1, \mu_1) = i_*(\mathcal{F}_2, \mu_2)$, that is to say, if $(\mathcal{F}_1, \mu_1) \sim_m (\mathcal{F}_2, \mu_2)$, then $U(\mathcal{F}_1, \mu_1)$ and $U(\mathcal{F}_2, \mu_2)$ are isotopic.

(c) We denote by $\beta U(\mathcal{F}, \mu)$ the union of of the boundary components of the support of $U(\mathcal{F}, \mu)$ which do not bound a disk in M. As an element of $\mathcal{S}'$, $\beta U(\mathcal{F}, \mu)$ does not depend on the measure class of $(\mathcal{F}, \mu)$.

In the spirit of 9.1, given by the fixed point theorem being considered, there are three possibilities:

i) $\beta U(\mathcal{F}, \mu) \neq \emptyset$

ii) $\beta U(\mathcal{F}, \mu) = \emptyset$ and $\lambda = 1$.

iii) $\beta U(\mathcal{F}, \mu) = \emptyset$ and $\lambda \neq 1$

In the rest of this exposé, we will analyse the three cases. We show that (i) is the "reducible" case, that (ii) is again a case of "finite order", whereas case (iii) is "pseudo-Anosov" (see exposé 1). The classification theorem is stated at the end of section 9.5. In this exposé, the surfaces are always orientable, but the diffeomorphisms do not necessarily preserve orientation, which complicates certain arguments, in particular lemma ??.

9.2 The reducible case: $\beta U(\mu) \neq \emptyset$

The relation refmeasure-equivalence implies that $U(\varphi(\mathcal{F}, \mu))$ and $U(\mathcal{F}, \lambda\mu)$ are isotopic. Hence, in $\mathcal{S}'$, we have the equality

$$\beta U(\mathcal{F}, \mu) = \beta(U(\mathcal{F}, \lambda\mu))$$

On the other hand,

$$\beta U(\varphi U(\mathcal{F}, \mu)) = \varphi(\beta U(\mathcal{F}, \mu))$$

$$\beta U(\varphi U(\mathcal{F}, \lambda\mu)) = \beta U(\mathcal{F}, \mu))$$

Hence, the *element* $\beta(\mathcal{F}, \mu)$ *of* S' *is invariant under* $[\varphi]$, possibly with the various components permuted.

In these conditions, φ is isotopic to a diffeomorphism φ' which leaves invariant the *submanifold* $\beta U(\mathcal{F}, \mu)$. By cutting M along this family of curves, we obtain a manifold (with boundary) W, possibly not connected, on which φ induces a diffeomorphism ψ. We recommence an analogous study of ψ by applying Thurston's theory of surfaces with boundary, which is sketched in exposé 11. Observe that W is simpler that M in the sense that every component of W has either smaller genus than M, or the same genus, but smaller Euler characteristic, in absolute value. Hence, in a finite number of stages we may give the structure of $\mathcal{F}$ up to isotopy.

9.3 Irrational Measured Foliations

irrational

By definition, a measured foliation $(\mathcal{F}, \mu)$ is **irrational** if $\beta U(\mathcal{F}, \mu)$ is empty.

Lemma 9.1 *(1) If* $(\mathcal{F}, \mu)$ *is an irrational measured foliation, the compact invariant set* X*, consisting of all singularities and the leaves joining any two singularities, does not possess any contractible connected components.*

(2) $\mathcal{F}$ *does not possess any closed, smooth leaves.*

Proof. The manifold $\overline{M - \operatorname{Supp} U(\mathcal{F}, \mu)}$ collapses onto X, [as in] (1). Suppose that Γ is a smooth leaf of $(\mathcal{F}, \mu)$; in fixing one of the sides of Γ in M and applying the stability lemma of exposé 5, one may find a maximal cylinder $\Phi \colon \Gamma \times [0,1] \to M$ such that

1) $\Phi(\Gamma \times \{0\}) = \Gamma$;
2) $\Phi(\Gamma \times [0,1])$ is therefore an embedding of the chosen side of Γ.

The genus being > 1, if the cylinder is maximal, then $\Phi(\Gamma \times \{1\}) \subset X$. In view of 1), the invariant set $\Phi \colon (\Gamma \times \{1\})$ is collapsible and we may show without difficulty that $\Phi \colon (\Gamma \times [0,1])$ is a disk D^2 with spine $\Phi(\Gamma \times \{1\})$. As there does not exist a measured foliation on D^2 such that ∂D^2 is a leaf, the existence of Γ is absurd. Hence, every half-leaf of $\mathcal{F}$ which does not have a singularity, is infinite. ∎

Remark. On the torus T^2, by the definition, every foliation is irrational, whereas a foliation which satisfies the conditions of lemma 1 is conjugate to a linear foliation with irrational slope.

Corollary 9.2 *Under the same conditions as in the preceding lemma, there exists $(\mathcal{F}', \mu')$, equivalent to $(\mathcal{F}, \mu)$, which does not possess any connections between singularities. This foliation is unique up to isotopy in its measure class.*

Proof. We obtain $(\mathcal{F}', \mu')$ by collapsing every component of the $\mathcal{F}$-invariant set X described above. The result of collapsing remains unchanged, up to isotopy, if we make a Whitehead smoothing on X. Uniqueness follows. ∎

Convention: In what follows, we will consistently represent a class of irrational foliations by the canonical model described above.

Lemma 9.3 *If $(\mathcal{F}, \mu)$ is the canonical model of a class of irrational measured foliations and if φ is a diffeomorphism such that $\varphi(\mathcal{F}, \mu) \sim_m \lambda(\mathcal{F}, \mu)$ for some λ in $\mathbb{R}_+^*$, then φ is isotopic to φ' such that*

$$\varphi'(\mathcal{F}, \mu) = (\mathcal{F}, \lambda\mu)$$

that is to say: φ' takes leaves to leaves and, for every arc α transverse to $\mathcal{F}$ we have

$$\mu({\varphi'}^{-1}) = \lambda\mu(\alpha).$$

Figure 9.1: Figure 0.1

Note: if $\lambda > 1$, this says that φ *contracts* the transverse distance (by a factor of $(1/\lambda)$), whereas if $\lambda < 1$, this says that φ *dilates* the transverse distance (by a factor of $(1/\lambda)$).

Proof. The foliations $\varphi(\mathcal{F}, \mu)$ and $(\mathcal{F}, \lambda\mu)$ are two canonical models of the same type; hence they are isotopic. By changing φ by this isotopy, one obtains the required φ'. ∎

$(\mathcal{F}, \mu)$-rectangle

Definition. Let $(\mathcal{F}, \mu)$ be any measured foliation. An $(\mathcal{F}, \mu)$**-rectangle** (or briefly, an $\mathcal{F}$-rectangle), is the image of an immersion $\varphi\colon [0,1] \times [0,1] \to M$ with the following properties:

(a) $\varphi\big|\,]0,1[\, \times\,]0,1[$ is a C^∞ embedding.

(b) $\varphi(\{t\} \times [0,1])$ is contained in a finite union of leaves and of singularities; if $t \in]0,1[$ then the image is contained in a single leaf.

(c) $\varphi([0,1] \times \{0\})$ and $\varphi([0,1] \times \{1\})$ are transverse to the leaves.

For an $\mathcal{F}$-rectangle R, I consider the decomposition $\partial R = \partial_{\mathcal{F}} R \cup \partial_\tau R$ where we define

$$\partial_{\mathcal{F}} R = \varphi(\{0,1\} \times [0,1]) \quad \text{and} \quad \partial_\tau R = \varphi([0,1] \times \{0,1\}).$$

I will denote by $\partial^0_{\mathcal{F}} R$ and $\partial^1_{\mathcal{F}} R$ the images respectively of $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$; an analogous notation will be used for $\partial_\tau R$. On the other hand, I will find it convenient to write $\operatorname{int} R = \varphi(]0,1[\times]0,1[)$, which in general is not the interior of the image; it is easy to see that $\operatorname{int} R$ and ∂R are disjoint.

good system of transversals

Definition. A **good system of transversals** for $\mathcal{F}$ is a finite system $\tau = \{\tau_i \mid i \in I\}$ of simple arcs with the following properties:

(a) every arc is transverse to $\mathcal{F}$ and may not meet a singularity at either of its endpoints;

(b) Two arcs do not meet at a single endpoint; if this is a singularity, the two arcs fall into two distinct sections.

Remark. One does not require that every arc contains a singularity.

Lemma 9.4 *Given a measured foliation $\mathcal{F}$ and a good system of transversals τ, there exists a system of rectangles $R_1, \ldots, R_N$, with the following properties.*

(1) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ for $i \neq j$.

(2) $\partial^{\epsilon}_{\tau} R_i$ is contained in a single arc of τ.

(3) Every $\partial^{\epsilon}_{\mathcal{F}} R_i$ contains a point of $\operatorname{Sing}(\mathcal{F}) \cup \partial\tau$; in other words, every rectangle R_i is maximal with respect to condition (2).

(4) The two sides of every arc of τ are covered by the rectangles.

The system $(R_1, \ldots R_N)$ is unique.

Remark. It is very instructive to take a small transversal to an irrational foliation of T^2 and to construct the corresponding rectangles.

Example: Cut the manifold as indicated in figure 0.3.

Proof. We obtain a manifold with boundary $\widehat{M}$ with a foliation $\mathcal{F}'$; the boundary τ' of $\widehat{M}$ is the "**dedouble**" of τ. Consider the finite set Z of τ', defined by the following conditions: dedouble

(1) $x \in \operatorname{Sing}(\mathcal{F}')$;

(2) x is one of the points giving an extremity of τ.

(3) The departing leaves of x abut a singularity of $\mathcal{F}$, where a point which gives an endpoint of τ.

By the Poincaré recurrence theorem (exposé 5), all leaves which depart from a point of $\tau' - Z$ return to $\tau' - Z$.

For every component α_i of $\tau' - Z$, by the maximal rectangle lemma (exposé 5), we may find a rectangle R_i such that $\partial^0_{\tau} R_i = \overline{\alpha_i}$ is the [attractor] of another component of $\tau' - Z$. When we view these in M, the rectangles are the desired rectangles. Uniqueness is left as an exercise. ∎

Lemma 9.5 *If, in the hypothesis of lemma 3, $\mathcal{F}$ is an irrational foliation, then*

$$R_1 \cup \cdots \cup R_N = M.$$

Proof. In any case, the union of the R_i is a closed $\mathcal{F}$-invariant set. If the boundary is not empty, there is a closed $\mathcal{F}$-invariant set consisting of cycles of leaves. If $\mathcal{F}$ is irrational, such a cycle cannot exist, hence the boundary is empty and $M = \bigcup R_i$. ∎

Lemma 9.6 *If $\mathcal{F}$ is an irrational foliation, every half-leaf L of $\mathcal{F}$ which does not lead to a singularity is dense.*

Proof. We claim that L is "infinite" (lemma 1). Let τ be a small arc transverse to $\mathcal{F}$ and $R_1, \ldots, R_N$ be the system of rectangles from lemma 3. By the above lemma, $\bigcup R_i = M$ and, since L is infinite, it contains the plaques in $\bigcup \operatorname{int} R_i$, since L meets τ. Since τ is arbitrary, L is dense. ∎

9.4 Case II: $(\mathcal{F}, \mu)$ is irrational and $\lambda = 1$.

Lemma 9.7 *If φ is a diffeomorphism and $(\mathcal{F}, \mu)$ is an irrational foliation such that*

$$\varphi(\mathcal{F}, \mu) = (\mathcal{F}, \mu)$$

then φ is isotopic to a diffeomorphism of finite order which preserves $(\mathcal{F}, \mu)$.

Proof. In the neighbourhood of every singularity, I chose transverse arcs, one in every sector, all of the same length with respect to the measure μ, as indicated in figure 0.4.

Seeing as $\lambda = 1$, we may choose the system of arcs τ in such a fashion that after an possible isotopy of φ, through diffeomorphisms which preserve $\mathcal{F}$, we have $\varphi(\tau) = \tau$.

Let $R_1, \ldots, R_N$ be the system of rectangles associated to τ (lemma 3). Since $\varphi(\tau) = \tau$ and $\varphi(\mathcal{F}) = \mathcal{F}$, we have that every $\varphi(R_i)$ is again an $\mathcal{F}$-rectangle satisfying condition (2) of lemma 3. It is easy to see that there exists a permutation π of $(1, \ldots, N)$ such that $\varphi(R_i) = R_{\pi(i)}$. In particular, φ transforms the graph $\Gamma = \bigcup_i \lambda \partial R_i$ in the same way. Hence φ permutes the edges of Γ among themselves. Working with the cycles of this permutation we may isotope φ to φ', by diffeomorphisms which preserve $\mathcal{F}$, such that $\varphi' \big| \Gamma$ is periodic and $\varphi'(R_i) = R_{\pi(i)}$.

Figure 9.2: Figure 0.2

Working with the cycles of π, we may make a second isotopy to obtain a periodic diffeomorphism, through diffeomorphisms which preserve $\mathcal{F}$. ∎

Remark. Such a diffeomorphism always has a fixed point in $\mathcal{T}M$.

Indeed, if φ is of finite order, φ' is an isometry in a certain metric m (whose curvature we cannot control); φ is hence an automorphism of the underlying conformal structure. Furthermore, by the uniformization theorem cited in exposé 7, underlying this structure there is a *unique* hyperbolic structure which, as a consequence, is invariant under φ.

9.5 Case III; $(\mathcal{F}, \mu)$ is irrational and $\lambda \neq 1$

I will suppose now that we are in the situation where $\varphi(\mathcal{F}, \mu) = (\mathcal{F}, \lambda\mu)$, with $\lambda \neq 1$, where $\mathcal{F}$ is a canonical model for a class of irrational foliations. By changing φ to φ^{-1} if necessary, we may assume that $\lambda > 1$.

Lemma 9.8 *The factor λ (respectively $1/\lambda$) is an algebraic integer of degree bounded by a quantity which is a function only of the genus of the surface.*

Proof. There is a branched covering $\widetilde{M}$ over M in which $(\mathcal{F}, \mu)$ lifts to a closed 1-form ω. If γ is a loop of $M - \mathrm{Sing}(\mathcal{F})$, along which $\mathcal{F}$ is orientable, then $\varphi(\gamma)$ has the same property; it follows that φ lifts to a diffeomorphism ψ of the open covering $\widetilde{M}\big|M - \mathrm{Sing}(\mathcal{F})$. This prolongs to a diffeomorphism $\widetilde{\psi}$ of $\widetilde{M}$. We have $(\widetilde{\varphi}^{-1})^*(\omega) = \lambda\omega$.

Hence λ is an eigenvalue of an automorphism of $H_1(\widetilde{M}, \mathbb{Z})$. Now, the rank of the cohomology group is bounded by a quantity which depends only on the genus of M. ∎

Lemma 9.9 *Under the hypothesis given above, [even if it entails] changing φ by an isotopy leaving $\mathcal{F}$ invariant, we may find a good system of transversals τ with the following properties.*

(1) In every sector of a singularity, there is an arc of τ. (figure 0.4).

(2) $\varphi(\tau) \subset r$, that is to say, φ takes every arc of τ into an arc of τ.

(3) If $x \in \partial\tau - \mathrm{Sing}(\mathcal{F})$, x belongs to the separatrices of a singularity; we denote by $\mathcal{F}_x$ the arc of the leaf joining x to $\mathrm{Sing}(\mathcal{F})$.

(4) Every separatrix contains an $\mathcal{F}_x$.

(5) $(\bigcup \mathcal{F}_x) \subset \varphi(\bigcup \mathcal{F}_x)$.

Proof. Since $\lambda > 1$, φ contracts the transversals (see the definition of the direct image of a measure). By modifying φ by an isotopy which preserves $\mathcal{F}$, is is easy to find a good system of transversals τ'' which satisfies (1) and (2) and which has the form of an arc in every sector. Let α'' be an arc of τ'' and L a separatrix issuing from a singularity s; in view of the density of the semi-leaves, there is, besides s, a first point of intersection of L with α''. Considering all of the separatrices, one obtains on α'' a finite number of such points; we subdivide α'' by these points, and shrink to a point the longest of the segments. ∎

Chapter 10

Some dynamics of pseudo-Anosov diffeomorphisms

by A. Fathi and M. Shub

We prove in this "exposé" that a pseudo-Anosov diffeomorphism realizes the minimum of topological entropy in its isotopy class. In section 10.1 we define topological entropy and give its elementary properties. In section 10.2 we define the growth of an endomorphism of a group and show that the topological entropy of a map is greater that the growth of the endomorphism it induces on the fundamental group. In section 10.3, we define subshifts of finite type and give some of their properties. In section 10.4, we prove that the topological entropy of a pseudo-Anosov diffeomorphism is the growth rate of the automorphism induced on the fundamental group, it is also $\log \lambda$, where $\lambda > 1$ is the stretching factor of f on the unstable foliation. In section 10.5, we prove the existence of a Markov partition for a pseudo-Anosov diffeomorphism, this fact is used in section 10.4. In section 10.6, we show that a pseudo-Anosov map is Bernoulli.

10.1 Topological Entropy

Topological entropy was defined to be a generalization of measure theoretic entropy [AKM65]. In some sense, entropy is a number (possibly infinite) which describes "how much" dynamics a map has. Here the emphasis, of course, must be on asymptotic behaviour. For example, if $f\colon X \to X$ is a map and $N_n(f)$ is the cardinality of the fixed point set of f^n, then $\limsup \frac{1}{n} \log N_n(f)$ is one measure of "how much" dynamics f has; but, if we consider $f \times R_\theta \colon X \times T^1 \to X \to T^1$ to be $(f \times R_\theta)(x, \alpha) = (f(x), \theta + \alpha)$ where $T^1 = \mathbb{R}/Z$ and θ is irrational then $N_n(f \times R_\theta) = 0$, and yet $f \times R_\theta$ should have at least as "much" dynamics as f. Topological entropy is a topological invariant which overcomes this difficulty.

We describe a lot of material frequently without crediting authors.

Definition. Let $f \colon X \to X$ be a continuous map of a compact topological space X. Let $\mathcal{A} = \{A_i\}_{i \in I}$ and $\mathcal{B} = \{B_j\}_{j \in J}$ be open covers of X. The open cover $\{A_i \cap B_j\}_{i \in I, j \in J}$ will be denoted by $\mathcal{A} \vee \mathcal{B}$. If $\mathcal{A}$ is a cover, $N_n(f, \mathcal{A})$ denotes the minimum cardinality of a subcover of $\mathcal{A} \vee f^{-1}\mathcal{A} \vee f^{-2}\mathcal{A} \vee \cdots \vee$ topological entropy $f^{-n+1}\mathcal{A}$, and $h(f, \mathcal{A}) = \limsup \frac{1}{n} \log N_n(f, \mathcal{A})$. The **topological entropy** of f is $h(f) = \sup_{\mathcal{A}} h(f, \mathcal{A})$ where the supremum is taken over all open covers of X.

Proposition 10.1 *Let X and Y be compact spaces. Let $f \colon X \to X, g \colon Y \to Y$ and $h \colon X \to Y$ be continuous. Suppose that h is surjective and $hf = gh$:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

then $h(f) \geq h(g)$.

In particular, if h is a homeomorphism, then $h(f) = h(g)$. So topological entropy is a topological invariant.

Proof. Pull back the open covers of Y to open covers of X. ∎

For metric spaces, compact or not, Bowen has proposed the following definition.

Definition. Suppose $f \colon X \to X$ is a continuous map of a metric space X and suppose $K \subset X$ is compact. Let ϵ be > 0. We say that a set $E \subset K$ is
(n, ϵ)-separated (n, ϵ)**-separated** if, given $x, y \in E$ with $x \neq y$, there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) \geq \epsilon$. We let $s_K(n, \epsilon)$ be the maximal cardinality of an (n, ϵ)-
(n, ϵ)-spanning separated set contained in K. We say that the set E is (n, ϵ)**-spanning** for K if, given $y \in K$, there is an $x \in E$ such that $d(f^i(x), f^i(y)) < \epsilon$ for each i with $0 \leq i < n$. We let $r_K(n, \epsilon)$ be the minimal cardinality of an (n, ϵ)-spanning

set contained in K. It is easy to see that $r_K(n,\epsilon) \leq s_K(n,\epsilon) \leq r_K(n,\epsilon/2)$. We let $\bar{s}_K(\epsilon) = \limsup \frac{1}{n} \log s_k(n,\epsilon)$ and $\bar{r}_K(\epsilon) = \limsup \frac{1}{n} \log r_k(n,\epsilon)$. Obviously $\bar{s}_K(\epsilon)$ and $\bar{r}_K(\epsilon)$ are decreasing functions of ϵ, and $\bar{r}_K(\epsilon) \leq \bar{s}_K(\epsilon) \leq \bar{r}_K(\epsilon/2)$. Hence, we may define $h_K(f) = \lim_{\epsilon\to 0} \bar{s}_K(\epsilon) = \lim_{\epsilon\to 0} \bar{r}_K(\epsilon)$. Finally, we put $h_X(f) = \sup\{h_K(f) \mid K\text{compact} \subset X\}$.

Proposition 10.2 *[Bow70, Din71]. If X is a compact metric space and $f\colon X \to X$ is continuous, then $h_X(f) = h(f)$.*

Proof. The proof is rather straightforward. By the Lebesgue covering lemma, every open cover has a refinement which consists of ϵ-balls. ▮

The number $h_X(f)$ depends on the metric on X and makes best sense for uniformly continuous maps.

Suppose that X and Y are metric spaces, we say that $p\colon X \to Y$ is a **metric covering map** if it is surjective and satisfies the following condition: there exists $\epsilon > 0$ such that, for any $0 < \delta < \epsilon$, any $y \in Y$ and any $x \in p^{-1}(y)$, the map $p\colon B_\delta(x) \to B_\delta(y)$ is a bijective isometry (here $B_\delta(\ )$ is the δ-ball). metric covering map

The main example we have in mind is the universal covering $p\colon \widetilde{M} \to M$ of a compact differentiable manifold M.

Proposition 10.3 *Suppose $p\colon X \to Y$ is a metric covering and $f\colon X \to X$, $g\colon Y \to Y$ are uniformly continuous. If $pf = gp$, then $h_X(f) = h_Y(g)$.*

Proof. It should be an easy estimate. The clue is that for $\ell > 0$ and for any sequence a_n we have $\limsup \frac{1}{n} \log(\ell a_n) = \limsup \frac{1}{n} \log a_n$. If $K \subset X$ and $K' \subset Y$ are compact and $p(K) = K'$, then there is a number $\ell > 0$ such that $\operatorname{card}(p^{-1}(y)) \leq \ell$ for all $y \in K'$. In fact, we may choose ℓ such that, if $\delta > 0$ is small enough, then $p^{-1}(B_\delta(y)) \cap K$ can be covered by at most ℓ 2δ-balls centered at points in $p^{-1}(B_\delta(y) \cap K$.

By the uniform continuity of f, we can find a δ_0 ($< \epsilon$) such that $x, x' \in X$ and $d(x,x') < \delta_0$ implies $d(f(x), f(x')) < \epsilon$, where $\epsilon > 0$ is the one given in the definition of a metric covering. If $2\delta < \delta_0$, it is easy to see that if $E' \subset K'$ is an (n,δ)-spanning set for g, then there exists an $(n,2\delta)$-spanning set $E \subset K$ for f, such that $\operatorname{card} E \leq \ell \operatorname{card} E'$. So, we have $r_K(n,2\delta) \leq \ell r_{K'}(n,\delta)$, hence $\bar{r}_K(f,2\delta) \leq \bar{r}_{K'}(g,\delta)$ and $h_K(f) \leq h_{K'}(g)$.

On the other hand, if $E \subset K$ is (n,η)-spanning (with $0 < \eta < \epsilon$) then $p(E) \subset K'$ is (n,η)-spanning. So $r_{K'}(n,\eta) \leq r_K(n,\eta)$, hence $h_{K'}(g) \leq h_K(f)$. Consequently $h_K(f) = h_{K'}(g)$. Since we sup over all compact sets and since p is surjective, we obtain $h_X(f) = h_Y(g)$. ∎

We add one additional fact.

Proposition 10.4 *If X is compact and $f\colon X \to X$ is a homeomorphism, then $h(f^n) = |n| h(f)$.*

For a proof, see [AKM65] or [Bow70].

10.2 The Fundamental Group

length

Given a finitely generated group G and a finite set of generators $\mathcal{G} = \{g_1, \dots, g_r\}$ of G, we define the **length** of an element g of G by $L_{\mathcal{G}}(g)$ = minimum length of a word in the g_i's and the g_i^{-1}'s representing the element g.

It is easy to see that if $\mathcal{G}' = \{g'_1, \dots, g'_s\}$ is another set of generators, then:

$$L_{\mathcal{G}}(g) \leq (\max L_{\mathcal{G}}(g'_i)) L_{\mathcal{G}'}(g)$$

If $A\colon G \to G$ is an endomorphism, let:

$$\gamma_A := \sup_{g \in G} \limsup \frac{1}{n} \log L_{\mathcal{G}}(A^n g) = \sup_{g_i \in \mathcal{G}} \limsup \frac{1}{n} \log L_{\mathcal{G}}(A^n g_i).$$

So γ_A is finite and by the inequality given above, γ_A does not depend on the set of generators.

Proposition 10.5 *If $A\colon G \to G$ is an endomorphism and $g \in G$, define $gAg^{-1}\colon G \to G$ by $[gAg^{-1}](x) = gA(x)g^{-1}$. We have $\gamma_A = \gamma_{gAg^{-1}}$.*

Caution: $(gAg^{-1})^n \neq gA^n g^{-1}$.

First, we need a lemma.

Lemma 10.6 *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences with a_n and $b_n \geq 0$ and k be > 0. We have:*

i) $\limsup \frac{1}{n} \log(a_n + b_n) = \max(\limsup \frac{1}{n} \log a_n, \limsup \frac{1}{n} \log b_n)$

ii) $\limsup \frac{1}{n} \log k a_n = \limsup \frac{1}{n} \log a_n$

iii) $\limsup \frac{1}{n} \log a_n \leq \limsup \frac{1}{n} \log(a_1 + \dots + a_n) \leq \max(0, \limsup \frac{1}{n} \log a_n)$

Proof. Put $a = \limsup \frac{1}{n} \log a$ and $b = \limsup \frac{1}{n} \log b_n$.
(i) The inequality $\max(a, b) \leq \limsup \frac{1}{n} \log(a_n + b_n)$ is clear.

If $c > \max(a, b)$, then we can find $n_0 \geq 1$ such that $n \geq n_0$ implies $a_n \leq e^{nc}$ and $b_n \leq e^{nc}$. We obtain for $n \geq n_0$:

$$\frac{1}{n} \log(a_n + b_n) \leq \frac{1}{n} \log(2e^{nc})$$

Hence $\limsup \frac{1}{n} \log(a_n + b_n) \leq \limsup \frac{1}{n} \log(2e^{nc}) = c$.
(ii) is clear.
(iii) The inequality $a \leq \limsup \frac{1}{n} \log(a_1 + \cdots + a_n)$ is clear.

Suppose $c > \max(0, a)$. We can find then $n_0 \geq 1$ such that $a_n \leq e^{nc}$ for $n \geq n_0$. We have for $n \geq n_0$:

$$a_1 + \cdots + a_n \leq \sum_{i=1}^{n_0-1} a_i + \frac{e^{(n+1-n_0)c} - 1}{e^c - 1} e^{n_0 c}.$$

It follows clearly that $\limsup_n \frac{1}{n} 1(a_1 + \cdots + a_n) \leq c$. ∎

Proof. (Proposition 10.5). If $x \in G$, we have:

$$(gAg^{-1})^n(x) = gA(g) \cdots A^{n-1}(g) A^n(x) A^{n-1}(g^{-1}) \cdots A(g^{-1}) g^{-1}$$

Suppose first that $A^{n_0}(g) = e$ for some n_0, then it is clear by lemma 10.6 (i) that:

$$\limsup \frac{1}{n} \log L_{\mathcal{G}}[(gAg^{-1})^n(x)] \leq \limsup \frac{1}{n} \log L_{\mathcal{G}}(A^n(x)).$$

If $A^n(g) \neq e$ for each $n \geq 1$, we have $L_{\mathcal{G}}(A^n(g)) \geq 1$, for each $n \geq 1$; hence $\limsup \frac{1}{n} \log L_{\mathcal{G}}(A^n(g)) \geq 0$. By lemma 10.6 (i) & (iii), we obtain:

$$\limsup \frac{1}{n} \log L_{\mathcal{G}}[(gAg^{-1})^n(x)] \leq \max(\limsup \frac{1}{n} \log L_{\mathcal{G}}((A^n(g)), \limsup \frac{1}{n} \log L_{\mathcal{G}}(A^n(x))).$$

This gives us $\gamma_{gAg^{-1}} \leq \gamma_A$, and by symmetry, we have $\gamma_{gAg^{-1}} = \gamma_A$. ∎

For a compact connected differentiable manifold, we interpret $\pi_1(M)$ as the group of covering transformations of the universal covering space $\widetilde{M}$ of M. If $f \colon M \to M$ is continuous, then there is a lifting $\tilde{f} \colon \widetilde{M} \to \widetilde{M}$. If $\tilde{f}_1$ and $\tilde{f}_2$ are both liftings of f, then $\tilde{f}_1 = \theta \tilde{f}_2$ for some covering transformation θ.

A given lifting $\tilde{f}_1$ determines an endomorphism $\tilde{f}_{1\#}$ of $\pi_1(M)$ by the formula $\tilde{f}_1\alpha = \tilde{f}_{1\#}(\alpha)\tilde{f}_1$ for any covering transformation α. If $\tilde{f}_1$ and $\tilde{f}_2$ are two liftings of f, then $\tilde{f}_1 = \theta\tilde{f}_2$ for some covering transformation θ and $\tilde{f}_1\alpha = \theta\tilde{f}_2\alpha = \theta\tilde{f}_{2\#}(\alpha)\tilde{f}_2 = \theta\tilde{f}_{2\#}(\alpha)\theta^{-1}\tilde{f}_1$, so $\tilde{f}_{1\#} = \theta\tilde{f}_{2\#}\theta^{-1}$ and $\gamma_{\tilde{f}_{1\#}} = \gamma_{\tilde{f}_{2\#}}$. Thus, we may define $\gamma_{f_\#} = \gamma_{\tilde{f}_\#}$ for any lifting $\tilde{f}\colon \widetilde{M} \to \widetilde{M}$ of f. If f has a fixed point $m_0 \in M$, then there is also a map $f_\#\colon \pi_1(M, m_0) \to \pi_1(M, m_0)$. The group $\pi_1(M, m_0)$ is isomorphic to the group of covering transformations of $\widetilde{M}$ and f may be lifted to $\tilde{f}$ such that $\tilde{f}_\#\colon \pi_1(M) \to \pi_1(M)$ is identified with $f_\#\colon \pi_1(M, m_0) \to \pi_1(M, m_0)$ by this isomorphism. Thus $\gamma_{f_\#}$ makes coherent sense in the case that f has a fixed point as well.

We suppose now that M has a Riemannian metric and we put on $\widetilde{M}$ a Riemannian metric by lifting the metric on M via the covering map $p\colon M \to M$. The map p is then a metric covering and the covering transformations are isometries. We have the following lemma due to Milnor [Mil68].

Lemma 10.7 *Fix $x_0 \in M$. There exist two constants $c_1, c_2 > 0$ such that for each $g \in \pi_1(M)$, we have:*

$$c_1 L_{\mathcal{G}}(g) \leq d(x_0, gx_0) \leq c_2 L_{\mathcal{G}}(g)$$

Proof. [Mil68]. Let $\delta = \operatorname{diam}(M)$, and define $N \subset \widetilde{M}$ by $N = \{x \in \widetilde{M} \mid d(x, x_0) \leq \delta\}$. We have $p(N) = M$. Remark that $\{gN\}_{g\in\pi_1(M)}$ is a locally finite covering of $\widetilde{M}$ by compact sets. Choose as a finite set of generators $\mathcal{G} = \{g \in \pi_1(M) \mid gN \cap N \neq \emptyset\}$ and notice that $g \in \mathcal{G} \iff g^{-1} \in \mathcal{G}$. Suppose $L_{\mathcal{G}}(g) = n$, then we can write $g = g_1 \cdots g_n$ with $g_i N \cap N \neq \emptyset$. It is easy to see then that $d(x_0, gx_0) \leq 2\delta n$. Hence, we obtain:

$$d(x_0, gx_0) \leq 2\delta L_{\mathcal{G}}(g)$$

Now, put $\nu = \min\{d(N, gN) \mid N \cap gN = \emptyset\}$, by compactness $\nu > 0$. Let k be the minimal integer such that $d(x_0, gx_0) < k\nu$. Along the minimizing geodesic from x_0 to gx_0, take $k+1$ points $y_0 = x_0, y_1, \ldots, y_{k-1}, y_k = gx_0$ such that $d(y_i, y_{i+1}) < \nu$ for $i = 0, \ldots, k-1$. Then, for $1 \leq i \leq k-1$, choose $y_i' \in N$ and $g_i \in G$ such that $y_i = g_i y_i'$ and put $g_0 = e$ and $g_k = g$. We have $d(g_i y_i', g_{i+1} y_{i+1}') < \nu$, hence $g_i^{-1} g_{i+1} \in \mathcal{G}$. From $g = (g_0^{-1} g_1) \cdots (g_{k-1}^{-1} g_k)$, we obtain $L_{\mathcal{G}}(g) < k$.

Since k is minimal, we have:

$$L_{\mathcal{G}}(g) \leq \frac{1}{\nu} d(x_0, gx_0) + 1 \leq (\frac{1}{\nu} + \frac{1}{\mu}) d(x_0, gx_0)$$

where $\mu = \min\{d(x_0, gx_0) \mid g \neq e, g \in \pi_1(M)\}$. ∎

Consider now $f\colon M \to M$ and let $\tilde{f}\colon \widetilde{M} \to \widetilde{M}$ be a lifting of f. Applying the lemma above, we obtain, for each $x_0 \in \widetilde{M}$:

$$\gamma_{f_\#} = \max_{g \in \pi_1(M)} \limsup \frac{1}{n} \log d(x_0, \tilde{f}^n_\#(g) x_0).$$

We next prove the following lemma:

Lemma 10.8 *Given $x, y \in \widetilde{M}$, we have:*

$$\limsup \frac{1}{n} \log d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq h(f).$$

Proof. Choose an arc α from x to y. If $y_1, \ldots, y_\ell \in \alpha$ is $(n+1, \epsilon)$-spanning for α and $\tilde{f}$, then $\tilde{f}^n(\alpha) \subset \cup_{i=1}^{\ell} B(\tilde{f}^n(y_i), \epsilon)$. Since $\tilde{f}^n(\alpha)$ is connected, this implies $\operatorname{diam}(\tilde{f}^n(\alpha)) < 2\epsilon\ell$. Hence:

$$d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq 2\epsilon\ell.$$

By taking ℓ to be minimal, we obtain

$$d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq 2\epsilon r_\alpha(n+1, \epsilon).$$

From this, we get:

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log d(\tilde{f}^n(x), \tilde{f}^n(y)) &\leq \limsup_{n\to\infty} \frac{1}{n} \log[2\epsilon r_\alpha(n+1, \epsilon)] = \bar{r}_\alpha(\epsilon) \\ &\leq h_\alpha(\tilde{f}) \leq h(\tilde{f}) = h(f). \end{aligned}$$

∎

We are now ready to prove:

Theorem 10.9 *If $f\colon M \to M$ is a continuous map, then:*

$$h(f) \geq \gamma_{f_\#}.$$

Proof. Since $\gamma_{f_\#} = \max\limits_{g\in\pi_1(M)}[\limsup \frac{1}{n}\log d(x_0, \tilde f^n_\#(g)x_0]$, we have to prove that for each $g \in \pi_1(M)$,

$$\limsup \frac{1}{n}\log d(x_0, \tilde f^n_\#(g)x_0) \le h(f).$$

We have:

$$d(x_0, \tilde f^n_\#(g)x_0) \le d(x_0, \tilde f^n(x_0)) + d(\tilde f^n(x_0), \tilde f^n_\#(g)\tilde f^n(x_0)) + d(\tilde f^n_\#(g)\tilde f^n(x_0), \tilde f^n_\#(g)x_0).$$

Since $\tilde f_\#(g)\tilde f^n = \tilde f^n g$, and the covering transformations are isometries, we obtain:

$$d(x_0, \tilde f^n_\#(g)x_0) \le 2d(x_0, \tilde f^n(x_0)) + d(\tilde f^n(x_0), \tilde f^n(gx_0)).$$

Remark also that:

$$d(x_0, \tilde f(x_0)) \le d(x_0, \tilde f(x_0)) + d(\tilde f(x_0), \tilde f^2(x_0)) + \cdots + d(\tilde f^{n-1}(x_0), \tilde f^n(x_0)).$$

By applying lemma 10.8 and lemma 10.6 (together with the fact $h(f) > 0$), we obtain:

$$\limsup \frac{1}{n}\log d(x_0, \tilde f_\#(g)x_0) \le h(f).$$

$\blacksquare$

The proof of the following lemma is straightforward.

Lemma 10.10 *If G_1 and G_2 are finitely generated groups, if $A\colon G_1 \to G_1$, $B\colon G_2 \to G_2$ and $p\colon G_1 \to G_2$ are homomorphisms with p surjective and $pA = Bp$:*

$$\begin{tikzcd} G_1 \arrow[r, "p"] \arrow[d, "A"] & G_2 \arrow[r] \arrow[d, "B"] & 0 \\ G_1 \arrow[r, "p"] & G_2 \arrow[r] & 0 \end{tikzcd}$$

then, $\gamma_A \ge \gamma_B$.

Applying this lemma to the fundamental group of M mod the commutator sub-group, we have

$$\begin{tikzcd} \pi_1(M) \arrow[r, "p"] \arrow[d, "f_\#"] & H_1(M) \arrow[r] \arrow[d, "f_{1*}"] & 0 \\ \pi_1(M) \arrow[r, "p"] & H_1(M) \arrow[r] & 0 \end{tikzcd}$$

so we obtain Manning's theorem [Man75].

Theorem 10.11 *If $f\colon M \to M$ is continuous, then $h(f) \geq \gamma_{f_{1*}} = \max \log \lambda$, where λ ranges over the eigenvalues of f_{1*}.*

Remark 1. For $\alpha \in \pi_1(M, m_0)$, we denote by $[\alpha]$ the class of loops freely homotopic to α. If M has a Riemannian metric, let $\ell([\alpha])$ be the minimum length of a (smooth) loop in this class. If $f\colon M \to M$ is continuous, $f[\alpha]$ is clearly well defined as a free homotopy class of loops. Let $G_f([\alpha]) = \limsup_n \frac{1}{n} \log[\ell(f^n[\alpha])]$ and let $G_f = \sup_\alpha G_f([\alpha])$.

It is not difficult to see that $G_f \leq \gamma_{f_\#}$. In fact, we have $\ell(f^n[\alpha]) \leq d(x_0, \tilde{f}^n_\#(\alpha)x_0]$, since the minimizing geodesic from x_0 to $\tilde{f}^n_\#(\alpha)x_0$ has an image in M which represents $f^n[\alpha]$.

Remark 2. It occurred to various people that Manning's theorem is a theorem about π_1. Among these are Bowen, Gromov and Shub. Manning's proof can be adapted. The proof above is more like Gromov [Gro] or Bowen [Bow70], but we take responsibility for any error. At first, we assumed that f had a periodic point or we worked with G_f. After reading Bowen's proof [Bow78], we eliminated the necessity for a periodic point.

Remark 3. If $x \in M$ and ρ is a path joining x to $f(x)$, we call $\rho_\#$ the homomorphism $\pi_1(M, f(x)) \to \pi_1(M, x)$. Since $f_\#\colon \pi_1(M, x) \to \pi_1(M, f(x))$, the composition

$$\rho_\# f_\#\colon [\gamma] \mapsto [\rho^{-1}\gamma\rho]$$

is a homomorphism of $\pi_1(X, x)$ into itself. This homomorphism can be identified with $\tilde{f}_\#$ for a lifting $\tilde{f}$ of f. Thus our result is the same as Bowen's [Bow78].

10.3 Subshifts of finite type

Let $A = (a_{ij})$ be a $k \times k$ matrix such that $a_{ij} = 0$ or 1, for $1 \leq i, j \leq k$, that is A is a $0 - 1$ matrix. Such a matrix A determines a subshift of finite type as follows. Let $S_k = \{1, \ldots, k\}$ and let $\Sigma(k) = \prod_{i=-\infty}^{i=\infty} S_k^i$, where $S_k^i = S_k$ for each $i \in \mathbb{Z}$. We put on S_k the discrete topology and on $\Sigma(k)$ the product topology. The subset $\Sigma_A \subset \Sigma(k)$ is the closed subset consisting of those bi-infinite sequences $\underline{b} = (b_n)_{n\in\mathbb{Z}}$ such that $a_{b_i b_{i+1}} = 1$ for all $i \in \mathbb{Z}$.

Pictorially, we imagine k boxes $1, 2, \dots, k$ and a point which at discrete "time n" can be in any one of the boxes. The bi-infinite sequences represent all possible histories of points. If we add the restriction that a point may move from box i to box j, if and only if $a_{ij} = 1$, then the set of all possible histories is precisely Σ_A.

shift

The **shift** $\sigma_A\colon \Sigma_A \to \Sigma_A$ is defined by $\sigma_A[(b_n)_{n\in\mathbb{Z}}] = (b'_n)_{n\in\mathbb{Z}}$ where $b'_n = b_{n+1}$ for each $n \in \mathbb{Z}$. Clearly, σ_A is continuous. Let $C_i \subset \Sigma(k)$ be defined by $C_i := \{\underline{x} \in \Sigma(k) \mid x_0 = i\}$. Let $D_i = C_i \cap \Sigma_A$, then $\mathcal{D} = \{D_1, \dots, D_k\}$ is an open cover of Σ_A by pairwise disjoint elements. For any $k \times k$ matrix $B = (b_{ij})$, we define the norm $||B||$ of B by $||B|| := \sum_{i,j=1}^k |b_{ij}|$. It is easy to see that $N_n(\sigma_A, \mathcal{D}) = \min \operatorname{card}(\mathcal{D} \vee \cdots \vee \sigma_A^{-n+1}\mathcal{D}) \le ||A^{n-1}||$ because the integer $a_{ij}^{(n)}$ is equal to the number of sequences $(i_0, \dots, i_n)$ with $i_\ell \in \{1, \dots, k\}, i_0 = i, i_n = j$ and $a_{i_\ell i_{\ell+1}} = 1$. So $\limsup \frac{1}{n} \log(N_n(\sigma_A, \mathcal{D})) \le \limsup \frac{1}{n} \log ||A^{n-1}|| = \limsup \log ||A^n||^{1/n}$. This latter number is recognizable as $\log$(spectral radius A) or $\log \lambda$, where λ is the largest modulus of an eigenvalue of A. In fact, we have:

Proposition 10.12 *For any subshift of finite type $\sigma_A\colon \Sigma_A \to \Sigma_A$, we have $h(\sigma_A) = \log \lambda$, where λ is the spectral radius of A.*

Proof. We begin by noticing that each open cover $\mathcal{U}$ of Σ_A is refined by a cover of the form $\bigvee_{i=-\ell}^{\ell} \sigma_A^{-i}\mathcal{D}$. This implies, with the notations of section 10.1:

$$N_n(\sigma_A, \mathcal{U}) \le \operatorname{card}\Big(\bigvee_{j=-\ell}^{n+\ell} \sigma_A^{-j}\mathcal{D}\Big) = \operatorname{card}\Big(\bigvee_{j=0}^{n+2\ell} \sigma_A^{-j}\mathcal{D}\Big) = N_{n+2\ell+1}(\sigma_A, \mathcal{D})$$

Hence, we obtain: $h(\sigma_A, \mathcal{U}) \le h(\sigma_A, \mathcal{D})$. This shows that $h(\sigma_A) = h(\sigma_A, \mathcal{D})$.

We now compute $h(\sigma_A, \mathcal{D})$. We distinguish two cases.

First case. Each state $i = 1, \dots, k$ occurs. This means that $D_i \neq \emptyset$ for each $D_i \in \mathcal{D}$. It is not difficult to show by induction that we have in fact

$$N_{n+1}(\sigma_A, \mathcal{D}) = \operatorname{card}(\mathcal{D} \vee \cdots \vee \sigma_A^{-n}\mathcal{D}) = \|A^n\|.$$

This proves the proposition in this case, as we saw above.

Second case. Some states do not occur. One can see that a state i occurs, if, and only if, for each $n \geq 0$, we have:

$$\sum_{j=1}^{k} a_{ij}^{(n)} > 0 \quad \text{and} \quad \sum_{j=1}^{k} a_{ji}^{(n)} > 0$$

where $A^n = (a_{ij}^{(n)})$.

Notice that if $\sum_{j=1}^{k} a_{ij}^{(n_0)} = 0$ then $\sum_{j=1}^{k} a_{ij}^{(n)} = 0$ for all $n \geq n_0$. This is because each $a_{\ell m}$ is ≥ 0.

Now, we partition $\{1, \ldots, k\}$ into three subsets X, Y, Z, where:

$$\begin{aligned} X &= \{i \mid \forall n \geq 0 \sum_{j=1}^{k} a_{ij}^{(n)} > 0 \quad \text{and} \quad \sum_{j=1}^{k} a_{ji}^{(n)} > 0\} \\ Y &= \{i \mid \exists n > 0 \sum_{j=1}^{k} a_{ij}^{(n)} = 0\} \quad = \quad \{i \mid \text{for } n \text{ large} \sum_{j=1}^{k} a_{ij}^{(n)} = 0\} \\ Z &= \{1, \ldots, k\} - (X \cup Y). \end{aligned}$$

We have:

$$Z \subset \{i \mid \text{for } n \text{ large} \sum_{j=1}^{k} a_{ji}^{(n)} = 0\}$$

By performing a permutation of $\{1, \ldots, k\}$, we can suppose that we have the following situation:

$$\{\underbrace{1, \ldots, t}_{X}, \underbrace{t+1, \ldots, s}_{Y}, \underbrace{s+1, \ldots, k}_{Z}\}$$

If B is a $k \times k$ matrix, we write:

$$B = \begin{bmatrix} B_{XX} & B_{XY} & B_{XZ} \\ B_{YX} & B_{YY} & B_{YZ} \\ B_{ZX} & B_{ZY} & B_{ZZ} \end{bmatrix}$$

where B_{KL} corresponds to the subblock of B having row indices in K and column indices in L.

It is easy to show that:

$$N_{n+1}(\sigma_A, \mathcal{D}) = \operatorname{card}(\mathcal{D} \vee \cdots \vee \sigma_A^{-n}\mathcal{D}) = \|A_{X,X}^n\|.$$

On the other hand, by the definition of Y and Z, for n large, A^n has the form:

$$A^n = \begin{bmatrix} (A^n)_{X,X} & (A^n)_{X,Y} & 0 \\ 0 & 0 & 0 \\ (A^n)_{Z,X} & (A^n)_{Z,Y} & 0 \end{bmatrix}$$

This implies that for n large, A^n and $(A^n)_{X,X}$ have the same non-zero eigenvalues, in particular

$$\log(\text{spectral radius} A^n_{X,X}) = n \log \lambda.$$

Remark also that we get, for n large and $k \geq 1$:

$$(A^{kn})_{X,X} = [(A^n)_{X,X}]^k.$$

This gives us, for n large:

$$\limsup_{k\to\infty} \frac{1}{kn+1} \log N_{kn+1}(\sigma_A, \mathcal{D}) = \limsup_{k\to\infty} \frac{1}{kn+1} \left\| [(A^n)_{X,X}]^k \right\| = \log \lambda$$

This implies that:

$$\log \lambda \leq h(\sigma_A, \mathcal{D}) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(\sigma_A, \mathcal{D})$$

As we showed the reverse inequality, we have:

$$\log \lambda = h(\sigma_A, \mathcal{D}) = h(\sigma_A).$$

∎

10.4 The entropy of pseudo-Anosov diffeomorphisms

Now we suppose that we have a compact, connected 2-manifold M without boundary with genus ≥ 2, and a pseudo-Anosov diffeomorphism $f \colon M \to M$. Hence there exists a pair $(\mathcal{F}^U, \mu^U)$ and $(\mathcal{F}^S, \mu^S)$ of transverse measured foliations with (the same) singularities such that $f(\mathcal{F}^S, \mu^S) = (\mathcal{F}^S, \frac{1}{\lambda}\mu^S)$ and $f(\mathcal{F}^U, \mu^U) = (\mathcal{F}^U, \lambda\mu^U)$ where $\lambda > 1$. This means, in particular, that f preserves the two foliations $\mathcal{F}^S$ and $\mathcal{F}^U$; it contracts the leaves of $\mathcal{F}^S$ by $\frac{1}{\lambda}$ and it expands the leaves of $\mathcal{F}^U$ by λ.

Let us recall that for any non-trivial simple closed curve α we have $\log\lambda = G_f(a)$ (see proposition **??**), hence we get $\log\lambda \le G_f$. [For the definition of G_f, look at the end of section 10.2.]

Proposition 10.13 *If $f\colon M \to M$ is pseudo-Anosov, then $h(f) = \gamma_{f_\#}$. So in particular, f has the minimal entropy of anything in its homotopy class. Moreover $h(f) = \log\lambda$ where λ is the expanding factor of f.*

Proof. Since $G_f \ge \log\lambda$, it suffices to show that $h(f) \le \log\lambda$ for a pseudo-Anosov diffeomorphism f. To do this, we find a subshift of finite type $\sigma_A\colon \Sigma_A \to \Sigma_A$ and a surjective continuous map $\Sigma_A \to M$ such that:

$$\begin{tikzcd} \Sigma_A \arrow[r, "\sigma_A"] \arrow[d, "\theta"] & \Sigma_A \arrow[d, "\theta"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

and $\log(\text{spectral radius} A) = h(\sigma_A) = \log\lambda$ for this same λ. Thus we will have $\log\lambda \le G_f \le \gamma_{f_\#} \le h(f) \le h(\sigma_A)$ or $\log\lambda \le h(f) \le \log\lambda$. ∎

In the following, we construct A and θ via Markov partitions.

First some definitions.

Definition. (compare chapter 8). A subset R of M is called a $(\mathcal{F}^S, \mathcal{F}^U)$**-rectangle**, or **birectangle**, if there exists an immersion $\phi\colon [0,1]\times[0,1] \to M$ whose image is R and such that: $(\mathcal{F}^S, \mathcal{F}^U)$-rectangle birectangle

- $\varphi\big|]0,1[\times]0,1[$ is an embedding;
- $\forall t \in [0,1], \varphi(\{t\}\times[0,1])$ is included in a finite union of leaves and singularities of $\mathcal{F}^S$, and in fact in one leaf if $t \in]0,1[$
- $\forall t \in [0,1], \varphi([0,1]\times\{t\})$ is included in a finite union of leaves and singularities of $\mathcal{F}^U$, and in fact in one leaf if $t \in]0,1[$.

We adopt the following notations: $\operatorname{int} R = \phi\big(]0,1[\times]0,1[\big)$, $\partial^0_{\mathcal{F}^S} R = \phi(\{0\}\times[0,1])$, $\partial^1_{\mathcal{F}^S} R = \phi(\{1\}\times[0,1])$, $\partial_{\mathcal{F}^S} R = \partial^0_{\mathcal{F}^S} R \cup \partial^1_{\mathcal{F}^S} R$ and in the same way, we define $\partial^0_{\mathcal{F}^U} R, \partial^1_{\mathcal{F}^U} R, \partial_{\mathcal{F}^U} R$.

Figure 10.1:

Remark that $\operatorname{int} R$ is disjoint from $\partial_{\mathcal{F}^S} R \cup \partial_{\mathcal{F}^U} R$, because $\phi\Big|]0,1[\times]0,1[$ is an embedding.

We call a set of the form $\phi(\{t\} \times [0,1])$ (resp. $\phi([0,1] \times \{t\})$) a $\mathcal{F}^S$-fiber, (resp. a $\mathcal{F}^U$-fiber) of R. We will call a birectangle **good** if ϕ is an embedding.

birectangle, good

If R is good birectangle, a point x of R is contained in only one $\mathcal{F}^S$-fiber which we will denote by $\mathcal{F}^S(x, R)$. In the same way, we define $\mathcal{F}^U(x, R)$.

Remark 1. If R is a $\mathcal{F}^U$-rectangle (see chapter 8) and $\partial^0_\tau R$ and $\partial^1_\tau R$ are contained in a union of $\mathcal{F}^S$-leaves and singularities, it is easy to see that R is in fact a birectangle.

Remark 2. We used the word birectangle instead of rectangle, even though rectangle is the standard word in Markov partitions, because this word was already used in chapter 8.

Remark 3. If R_1 and R_2 are birectangles and $R_1 \cap R_2 \neq \emptyset$ then it is a finite union of birectangles and possibly of some arcs contained in $(\partial_{\mathcal{F}^S} R \cup \partial_{\mathcal{F}^U} R) \cap (\partial_{\mathcal{F}^S} R \cup \partial_{\mathcal{F}^U} R)$.

Moreover the birectangles are the closures of the connected components of $\operatorname{int} R_1 \cap \operatorname{int} R_2$.

width

If R is a birectangle, we define the **width** of R by:

$$\mathcal{W}(R) = \max\{\mu^U(\mathcal{F}^S\text{-fiber}), \mu^S(\mathcal{F}^U\text{-fiber})\}.$$

Lemma 10.14 *There exists $\epsilon > 0$ such that, if R is a birectangle with $\mathcal{W}(R) \leq \epsilon$, then it is a good rectangle.*

Proof. [Sketch] If a birectangle is contained in a coordinate chart of the foliations, then it is automatically a good birectangle. The existence of ϵ follows from compactness. ∎

Lemma 10.15 *There exists $e > 0$ such that if α (resp. β) is an arc contained in a finite union of leaves and singularities of $\mathcal{F}^S$ (resp. $\mathcal{F}^U$) with $\mu^U(\alpha) < \epsilon$ (resp. $\mu^S(\beta) < \epsilon$), then the intersection of α and β is at most one point.*

Definition. A **Markov partition** for the pseudo-Anosov diffeomorphism $f\colon M \to M$ is a collection of birectangles $R = \{R_1, \ldots, R_k,\}$ such that

Markov partition

1. $\bigcup_{i=1}^k R_i = M$;
2. R_i is a good rectangle;
3. $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ for $i \neq j$;
4. If x is in $\operatorname{int}(R_i)$ and $f(x)$ is in $\operatorname{int}(R_j)$, then
$$f(\mathcal{F}^S(x, R_i)) \subset \mathcal{F}^S(f(x), R_j), \quad \text{and} \quad f^{-1}(\mathcal{F}^U(f(x), R_j)) \subset \mathcal{F}^U(x, R_i);$$
5. If x is in $\operatorname{int}(R_i)$ and $f(x)$ is in $\operatorname{int}(R_j)$, then
$$f(\mathcal{F}^U(x, R_i)) \cap (R_j) = \mathcal{F}^U(f(x), R_j) \quad \text{and} \quad f^{-1}(\mathcal{F}^S(x, R_j)) \cap R_i = \mathcal{F}^S(x, R_i);$$

This means that $f(R_i)$ goes across R_j just one time.

We will show in next section how to construct a Markov partition for a pseudo-Anosov diffeomorphism.

Given a Markov partition $\mathcal{R} = (R_1, \ldots, R_k\}$, we construct the subshift of finite type Σ_A and the map $h\colon \Sigma_A \to M$ as follows. Let A be the $k \times k$ matrix defined by $a_{ij} = 1$ if $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \emptyset$, and $a_{ij} = 0$ otherwise. If $\underline{b} \in \Sigma_A$ then $\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is non-empty and consists in fact of a single point. This will follow from the following lemma.

Figure 10.2:

Lemma 10.16 *i) Suppose* $a_{ij} = 1$*, then* $f(R_i) \cap R_j$ *is a non-empty (good) birectangle which is a union of* $\mathcal{F}^U$*-fibers of* R_j*.*

ii) Suppose moreover that C *is a birectangle contained in* R_i *which is a union of* $\mathcal{F}^U$*-fibers of* R_i*, then* $f(C) \cap R_j$ *is a non-empty birectangle which is a union of* $\mathcal{F}^U$*-fibers of* R_j*.*

iii) Given $\underline{b} \in \Sigma_A$*, for each* $n \in \mathbb{N}$*,* $\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$ *is a non-empty birectangle. Moreover, we have* $\mathcal{W}(\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})) \leq \lambda^{-n} \max\{\mathcal{W}(R_1), \ldots, \mathcal{W}(R_k)\}$*.*

Proof. Since $a_{ij} = 1$, we can find $x \in \operatorname{int}(R_i) \cap f^{-1}(\operatorname{int} R_j)$. We have $f(\mathcal{F}^S(x, R_i)) \subset \mathcal{F}^S(f(x), R_j) \subset R_j$. Since each $\mathcal{F}^U$-fiber of R_i intersects $\mathcal{F}^S(x, R_i)$, we obtain that the image of each $\mathcal{F}^U$-fiber of R_i intersects R_j. Moreover, by condition 5), $f[R_i - \partial_{\mathcal{F}^U} R_i] \cap R_j$ is an union of $\mathcal{F}^U$-fibers of R_j, hence $f(R_i) \cap R_j = \overline{f(R_i - \partial_{\mathcal{F}^U} R_i) \cap R_j}$ is also a union of $\mathcal{F}^U$-fibers of R_j. This proves i). The proof of ii) is the same.

To prove iii), remark first that it follows by induction on n using ii) that each set of the form $f^n R_{b_i} \cap f^{n-1}(R_{b_{i+1}}) \cap \cdots \cap R_{b_{i+n}}$ is a non-empty birectangle which is a union of $\mathcal{F}^U$-fibers of $R_{b_{i+n}}$. In particular, $\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$ is a non-empty birectangle in R_{b_0}. The estimate of the width is clear. ∎

By the lemma, if $\underline{b} \in \Sigma_A$ the set $\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is the intersection of a decreasing sequence of non-empty compact sets, namely the sets $\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$ for $n \in \mathbb{N}$.

Hence $\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is non-void. It is reduced to one point because $\mathcal{W}(\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i}))$ tends to zero as n goes to infinity.

The map $\theta\colon \Sigma_A \to M$ given by $\theta(\underline{b}) = \bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i})$ is well defined, it is easy to see that it is continuous and that the following diagram commutes

$$\begin{tikzcd} \Sigma_A \arrow[r, "\sigma_A"] \arrow[d, "\theta"'] & \Sigma_A \arrow[d, "\theta"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

We show now that θ is surjective. First remark that, for each $i = 1, \dots, k$ the closure of $\mathrm{int}(R_i)$ is R_i. Hence $V = \bigcup_{i=1}^k \mathrm{int}(R_i)$ is a dense open set. By the Baire category theorem $U = \bigcap_{i \in \mathbb{Z}} f^{-i}(V)$ is dense in M. If $x \in U$, then for each $n \in \mathbb{Z}$, the point $f^n(x)$ is in a unique $\mathrm{int}(R_{b_n})$ and $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ is an element of Σ_A. It is clear that $\theta(\underline{b}) = x$. Thus $\theta(\Sigma_A) \supset U$. As Σ_A is compact and h continuous, we have $\theta(\Sigma_A) = M$.

Up to now, we have obtained that

$$\log \lambda \le G_f \le \gamma_{f_\#} \le h(f) \le h(\sigma_A) = \log(\text{spectral radius of } A)$$

All that remains is to show that:

$$(\text{spectral radius of } A) = \lambda.$$

To see this, we do the following thing. Put $y_i = \mu^U((\mathcal{F}^S\text{-fiber of } R_i))$, it is clear that this quantity is independent of the $\mathcal{F}^S$-fiber of R_i and also $y_i > 0$.

We have trivially the following equality:

$$y_j = \sum_{i=1}^k \frac{y_i}{\lambda} a_{ij}$$

which gives:

$$\lambda y_j = \sum_{i=1}^k y_i a_{ij}$$

[in particular λ is an eigenvalue of A.] Hence. we obtain

$$\lambda y_j \ge \Big(\sum_{i=1}^k a_{ij} \Big) \min_i y_i$$

This gives us:

$$\lambda \Big(\sum_j y_j \Big) \ge ||A|| \min_i y_i$$

where $|| \; ||$ is the norm introduced in section 10.3.

In the same way, we obtain for each $n \geq 2$:

$$\lambda^n \Big(\sum_j y_j \Big) \geq ||A^n|| \min_i y_i$$

Hence:

$$\lambda \geq ||A^n||^{1/n} \Big(\frac{\min(y_1, \ldots, y_k)}{\sum_j y_j} \Big)^{1/n}.$$

Since $\min(y_1, \ldots, y_k) > 0$,

$$\lim_{n \to \infty} \Big(\frac{\min(y_1, \ldots, y_k)}{\sum_j y_j} \Big)^{1/n} = 1$$

We thus obtain:

$$\lambda \geq \lim_{n \to \infty} ||A^n||^{1/n} = \text{spectral radius of } A.$$

Since λ is an eigenvalue of A, we obtain:

$$\lambda = \text{spectral radius of } A.$$

10.5 Construction of Markov partitions for pseudo-Anosov diffeomorphisms

In this section, we still consider $f \colon M \to M$ a pseudo-Anosov diffeomorphism and we keep the notations of the last section. We sketch the proof of the following proposition.

Proposition 10.17 *A pseudo-Anosov diffeomorphism has a Markov partition.*

Proof. Using the methods given in section **??**, it is easy, starting with a family of transversals to $\mathcal{F}^U$ contained in $\mathcal{F}^S$-leaves and singularities, to construct a family $\mathcal{R}$ of $\mathcal{F}^U$-rectangles $R_1, \ldots, R_\ell$ such that

(i) $\bigcup_{i=1}^{\ell} R_i = M$;

(ii) $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $i \neq j$;

(iii) $f^{-1}(\bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^U} R_i) \subset \bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^U} R_i$, $f(\bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^S} R_i) \subset \bigcup_{i=1}^{\ell} \partial_{\mathcal{F}^S} R_i$, By the remark following the definition of birectangles, the R_i's are birectangles since the system of transversals is contained in $\mathcal{F}^S$-leaves and singularities.

We define for each n a family of birectangles $\{\mathcal{R}_n\}$ in the following way: the birectangles of $\{\mathcal{R}_n\}$ will be the closures of the connected components of the non-empty open sets contained in

$$\bigvee_{i=-n}^{n} f^i R^\circ = \left\{ \bigcap_{i=-n}^{n} f^i(\operatorname{int} R_{a_i}) \mid R_{a_i} \in \mathcal{R} \right\}.$$

It is easy to see that $\mathcal{R}_n$ still satisfies the properties (i), (ii) and (iii) given above. Moreover, if $R \in \mathcal{R}_n$, we have $\mathcal{W}(R) \leq \lambda^{-n} \max\{\mathcal{W}(R_i) \mid R_i \in \mathcal{R}\}$. In particular, by lemma 10.14 of last section, for n sufficiently large, each birectangle R in $\mathcal{R}_n$ is a good one.

We assert that for n sufficiently large $\mathcal{R}_n$ is a Markov partition. All that remains is to verify properties (4) and (5) of a Markov partition. It is an easy exercise to show that property (4) is a consequence of property (iii) given above (see lemma **??**). By lemma 10.15, if n is sufficiently large and $R, R' \in \mathcal{R}_n$, then if $x \in R$, $f(\mathcal{F}^U(x, R))$ intersects in at most one point each $\mathcal{F}^S$-fiber of R'. Property (5) follows easily from the combination of this fact and of property (4). ∎

Example of Markov partition on T. Let $A\colon T^2 \to T^2$ be the linear map defined by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Here $T^2 = \mathbb{R}^2/\mathbb{Z}^2$; and A acts on $\mathbb{R}$ preserving $\mathbb{Z}$, thus A defines a map of T. The translates of the eigenspaces of A foliate T. The map A on T is Anosov. The foliation of T corresponding to the eigenvalue $\frac{3+\sqrt{5}}{2}$ is expanded, the foliation corresponding to $\frac{3-\sqrt{5}}{2}$ is contracted.

We draw a fundamental domain with eigenspaces approximately drawn in (see figure 10.3.)

The endpoints of the short stable manifold are on the unstable manifolds after equivalences have been made. Filling in to maximal rectangles gives us the picture in figure 10.4.

Figure 10.3:

Figure 10.4:

The hatched line is the extension of the unstable manifold. Identified pieces are numbered similarly. One rectangle is given by 1,2,3,6 and the other by 4,5. This partition in two rectangles gives a Markov partition by taking intersections with direct and inverse images.

The construction of the Markov partition of a pseudo-Anosov diffeomorphism $f\colon M$, which preserves orientation and fixes the prongs of $\mathcal{F}^S$ and $\mathcal{F}^U$, is the same as in the example above. We sketch here the argument, hoping that it will aid the reader to understand the general case.

Since the unstable prongs are dense, we may pick small stable prongs whose endpoints lie on unstable prongs.

Roughly, the picture is as shown in figure 10.5.

We may extend these curves to maximal birectangles leaving the drawn curves as boundaries. By density of the leaves, every leaf crosses a small stable prong, so the rectangles obtained this way cover M^2. The extension process requires that the unstable prongs be extended perhaps but the extension remains connected. Thus we have a partition by birectangles with boundaries the unions of connected segments lying on stable or unstable prongs. Consequently an unstable leaf entering the interior of a birectangle under f can't end in the interior, because the stable boundary has been taken to the stable

Figure 10.5: Small stable prong

boundary, etc…

Figure 10.6:

The only thing left is to make the partition sufficiently small. To do this, it is sufficient to take the birectangles obtained by intersections $f^{-n}\mathcal{R} \vee \cdots \vee \mathcal{R} \vee \cdots f^n(\mathcal{R})$ for n sufficiently large.

10.6 Pseudo-Anosov diffeomorphisms are Bernoulli

A pseudo-Anosov diffeomorphism $f \colon M \to M$ has a natural invariant probability measure μ which is given locally by the product of μ^S restricted to plaques of $\mathcal{F}^S$ with μ^U restricted to plaques of $\mathcal{F}^U$. The goal of this section is to sketch the proof of the following theorem.

Theorem 10.18 *The dynamical system (M, f, μ) is isomorphic (in the measure theoretical sense) to a Bernoulli shift.*

Recall that a Bernoulli shift is a shift $(\Sigma(\ell), \sigma)$ together a measure ν which is the infinite product of some probability measure on $\{1, \dots, \ell\}$. Obviously, ν is invariant under σ, see [Orn74], [Sin76b].

We will have to use the notion and properties of measure theoretic entropy, see [Sin76b]. We will also need the following two theorems on subshifts of finite type.

Let A be a $k \times k$ matrix and (Σ, σ_A) be the subshift of finite type obtained from it.

Theorem 10.19 (Parry [Par64]) *Suppose that A has all its entries > 0 for some n. Then, there is a probability measure ν_A invariant under σ_A such that the measure theoretic entropy $h_{\nu_A}(\sigma_A)$ is equal to the topological entropy $h(\sigma_A)$. Moreover, ν_A is the only invariant probability measure having this property, and $(\Sigma_A, \sigma_A, \nu_A)$ is a mixing Markov process.*

Theorem 10.20 (Friedman-Ornstein [Orn74]) *A mixing Markov process is isomorphic to a Bernoulli shift. In particular, the $(\Sigma_A, \sigma_A, \nu_A)$ above is Bernoulli.*

Now we begin to prove that (M, f, μ) is Bernoulli. For this, we will use the subshift (Σ_A, σ_A) and the map $\theta\colon (\Sigma_A, \sigma_A) \to (M, f)$ obtained from the Markov partition $\mathcal{R} = \{R_1, \dots, R_k\}$.

Lemma 10.21 *There exists $n \geq 1$ such that A^n has entries > 0.*

Proof. Given R_i, we can find a periodic point $x_i \in R_i^\circ$, call n_i its period. Consider the unstable fiber $\mathcal{F}^U(x_i, R_i)$; we have, for $\ell \geq 0$, $f^{\ell n_i}(\mathcal{F}^U(x_i, R_i)) \subset \mathcal{F}^U(x_i, R_i)$.

Moreover the μ^S-length of $f^{\ell n_i}(\mathcal{F}^U(x_i, R_i))$ is $\lambda^{\ell n_i} \mu^S(\mathcal{F}^U(x_i, R_i))$. This implies that

$$f^{\ell n_i}(\mathcal{F}^U(x_i, R_i)) \cap R_j^\circ \neq \emptyset \quad \forall\, j = 1, \dots, k,$$

$n = \ell \cdot \prod_{i=1}^k n_i$ with ℓ large enough, we get $f(R_i) \cap R_j \neq \emptyset$ for each pair (i, j). Hence, we obtain that $a_{ij}^{(n)} > 0$ for each (i, j), where $A^n = (a_{ij}^{(n)})$ ▮

This lemma shows that $(\Sigma_A, \sigma_A, \nu_A)$ is Bernoulli by the results quoted above. All we have to do now is to prove that (M, f, μ) is isomorphic to $(\Sigma_A, \sigma_A, \nu_A)$.

Lemma 10.22 *The measure theoretic entropy $h_\mu(f)$ is $\log\lambda$.*

Proof. Since topological entropy is the supremum of measure theoretical entropies (see [Bow70], [Goo71]), we have $h_\mu(f) \le \log\lambda$. Consider now the partition $\mathcal{R}^\circ = \{\operatorname{int} R_i\}$; its μ-entropy $h_\mu(f, \mathcal{R}^\circ)$ with respect to f is given by:

$$h_\mu(f, \mathcal{R}^\circ) = \lim_n -\frac{1}{n}\sum a_{ij}^{(n)}\lambda^{-n}y_ix_j\log(\lambda^{-n}y_ix_j)$$

where $y_i = \mu^U(\mathcal{F}^S\text{-fiber of } R_i)$ and $x_j = \mu^S(\mathcal{F}^U\text{-fiber of } R_j)$. As we saw at the end of section 10.4, $\frac{a_{ij}^{(n)}}{\lambda^n} \le \frac{||A^n||}{\lambda^n}$ is bounded (by $\frac{\sum y_i}{\min y_i}$). This implies:

$$\lim_n -\frac{1}{n}\sum a_{ij}^{(n)}\lambda^{-n}y_ix_j\log y_ix_j = 0.$$

We have also:

$$\sum a_{ij}^{(n)}\lambda^{-n}y_ix_j = \sum y_jx_j = \sum \mu(R_j^\circ) = \mu(M) = 1$$

By putting these facts together, we obtain: $h_\mu(f, \mathcal{R}^\circ) = \log\lambda$. Hence, $h_\mu(f) = \log\lambda$, because $\log\lambda = h_\mu(f, \mathcal{R}^\circ) \le h_\mu(f) \le h(f) = \log\lambda$. ∎

Proof. Proof of the theorem. Put $\partial\mathcal{R} = \bigcup_{i=1}^k \partial R_i$, we have $\mu(\partial\mathcal{R}) = 0$. This implies that the set $Z = M - \bigcup_{i\in\mathbb{Z}} f^i(\partial R)$ has μ-measure equal to one. We know by section 10.4 that θ induces a (bi-continuous) bijection of $\theta^{-1}(Z)$ onto Z, we can then define a probability measure ν on Σ_A by $\nu(B) = \mu(\theta([\theta^{-1}(Z)\cap B])$ for each Borel set $B \subset \Sigma_A$. It is easy to see that ν is σ_A invariant; moreover, θ gives rise to a measure theoretic isomorphism between $(\Sigma_A, \sigma_A, \nu)$ and (M, f, μ). In particular $h_\nu(\sigma_A) = h_\mu(f) = \log\lambda$. Since $\log\lambda$ is also the topological entropy of σ_A we obtain from Parry's theorem that $\nu = \nu_A$ and that $(\Sigma_A, \sigma_A, \nu)$ is a mixing Markov process. By the Friedman-Ornstein theorem, $(\Sigma_A, \sigma_A, \nu)$ is Bernoulli, hence (M, f, μ) is also Bernoulli. ∎

Chapter 11

Thurson’s theory for Surfaces with Boundary

by F. Laudenbach

Chapter 12

Uniqueness theorems for pseudo-Anosov diffeomorphisms.

by A. Fathi and V. Poénaru

Chapter 13

Construction of pseudo-Anosov diffeomorphisms.

by F. Laudenbach

Chapter 14

Fibrations of S^1 with Pseudo-Anosov Monodromy

by David Fried

We will develop Thurston's description of the collection of fibrations of a closed three manifold over S^1. We will then show that the suspended flows of pseudo-Anosov diffeomorphisms are canonical representatives of their nonsingular homotopy class, thus extending Thurston's theorem for surface homeomorphisms to a class of three dimensional flows. Our proof uses Thurston's work on fibrations and surface homeomorphisms and our criterion for cross-sections to flows with Markov partitions. We thank Dennis Sullivan for introducing Thurston's results to us. We are also grateful to Albert Fathi, François Laudenbach and Michael Shub for their helpful suggestions.

A smooth fibration $f\colon X \to S^1$ of a manifold over the circle determines a nonsingular (i.e., never zero) closed 1-form $f^*(d\theta)$ with integral periods. Conversely if ω is a nonsingular closed 1-form and X is closed, then the map $f(x) = \int_{x_0}^{x} \omega$ from X to $\mathbb{R}/\text{periods}(\omega)$ will be a fibration over S^1 provided the periods of ω have rational ratios. For since $\pi_1 X$ is finitely generated, the periods of ω will be a cyclic subgroup of $\mathbb{R}$ (not trivial since X is compact and f open) and we have $\mathbb{R}/\text{periods} \cong S^1$. By constructing a smooth flow ψ on X with $\omega(\frac{d\psi}{dt}) = 1$, we see that f is a fibration. The relation of nonsingular closed 1-forms to fibrations over S^1 is very strong indeed, as the following theorem (which gives strong topological constraints on the existence of nonsingular closed 1-forms) indicates.

Theorem 14.1 ([Tis70]) *For a compact manifold X, the collection $\mathcal{C}$ of nonsingular* *classes, that is the cohomology classes of nonsingular closed 1-forms on X, is an open cone in $H^1(X;\mathbb{R}) - \{0\}$. The cone $\mathcal{C}$ is nonempty if and only if X fibers over S^1.*

Proof. The openness of $\mathcal{C}$ follows easily from de Rham's Theorem. If $\eta_1, \ldots, \eta_d$ are closed 1-forms that span $H^1(X;\mathbb{R})$ and if ω_0 is a closed 1-form, then the forms $\omega_a = \omega_o + \sum_{i=1}^{d} a_i\eta_i$, $|a_i| < \epsilon$, represent a neighborhood of $[\omega_0]$ in $H^1(X;\mathbb{R})$. If ω_0 is nonsingular and ϵ sufficiently small, then the ω_a are nonsingular. The forms $\lambda\omega_a$, with $\lambda > 0$ represent all positive multiples of $[\omega_a]$, so $\mathcal{C}$ is an open cone.

Choosing a so that the periods of ω_a are rationally related, we see that X fibers over S^1. We already noted that $0 \notin \mathcal{C}$. ∎

In dimension 3, Stallings characterized the elements of $\mathcal{C} \cap H^1(X;\mathbb{Z}) \subset H^1(X;\mathbb{R})$. We note that if X is closed, connected and oriented and does fiber over S^1 with fibers of positive genus, then X will be covered by Euclidean space $\mathbb{R}^3$. Thus X will be **irreducible**, that is, every sphere S^2 embedded in X must bound a ball (this follows from Alexander's theorem showing $\mathbb{R}^3$ is irreducible). *We assume hence-forward* that M is a closed, connected oriented and irreducible 3-dimensional manifold.

irreducible

Theorem 14.2 ([Sta61]) *If $u \in H^1(M;\mathbb{Z}) - \{0\}$, then there is a fibration $f\colon M \to S^1$ with $[f^*(d\theta)] = u$, if and only if $\ker(u : \pi_1 M \to \mathbb{Z})$ is finitely generated.*

We observe that the forward implication holds even for finite complexes since the homotopy exact sequence identifies the kernel as the fundamental group of the fiber.

Theorem 14.2 reduces the geometric problem of fibering M to an algebraic problem, with only two practical complications. First, whenever $\dim H^1(M;\mathbb{R}) > 1$, there are infinitely many u to check. Secondly, it is difficult to decide if $\ker u$ is finitely generated. An infinite presentation may be readily constructed by the Reidemeister-Schreier process; this yields an effective procedure for deciding if the abelianization of $\ker u$ is finitely generated (we work out an example of this at the end of the chapter.)

Thurston's theorem (theorem 14.6 below) helps to minimize the first problem and make Stalling's criterion more practical. It will be seen that one need only examine finitely many u, provided one can compute a certain natural seminorm on $H^1(M;\mathbb{R})$.

As $H^1(M;\mathbb{Z}) \subset H^1(M;\mathbb{R})$ is a lattice of maximal rank, the seminorm will be determined by its values on $H^1(M;\mathbb{Z})$. Each $u \in H^1(M;\mathbb{Z})$ is geometrically represented by framed surfaces under the Pontrjagin construction [Mil66]. A framed (that is, normally oriented) surface S represents u whenever there is a smooth map $f\colon M \to S^1$ with regular value x so that $S = f^{-1}(x)$ and $u = f^*(d\theta)$. By irreducibility of M, any framed sphere in M represents the 0 class so S may be taken **sphereless** (that is, all components of S have Euler characteristic ≤ 0.) sphereless

Definition. $\|u\| := \min\{-\chi(S) \mid S$ is a sphereless framed surface representing $u\}$ ∎

It is important to observe that a sphereless framed surface S in M, with $\|u\| = -\chi(S)$, must be **incompressible** (that is, for each component $S_i \subset S$, $\pi_1(S_i) \to \pi_1 M$ is injective.) For (see Kneser's lemma [Sta71]), one could otherwise attach a 2-handle to S_i so as to lower $-\chi(S)$ without introducing spherical components. incompressible

The justification for the notation $\|u\|$ is the following result.

Theorem 14.3 ([Thu86]) *$\|u\|$ is a seminorm on $H^1(M;\mathbb{Z})$.*

This follows from standard 3-manifold techniques. The triangle inequality follows from the incompressibility of minimal representatives and some cut-and-paste arguments. The homogeneity follows by the covering homotopy theorem for the cover $z^n\colon S^1 \to S^1$.

One instance where $\|u\|$ is easily computed is where u is represented by the fiber K of a fibration $f\colon M \to S^1$. We have:

Proposition 14.4 ([Thu86]) *If $K \to M \xrightarrow{f} S^1$ is a fibration, then $\big\|[f^*(d\theta)]\big\| = -\chi(K)$.*

Proof. By homogeneity we may suppose that $u = [f^*(d\theta)]$ is indivisible, that is $u(\pi_1 M) = \pi_1 S^1$. This implies that K is connected and that $K \times \mathbb{R}$ is the infinite cyclic cover of M determined by u. If K is a torus we are done, so assume $-\chi(K) > 0$. Any sphereless framed surface S representing u lifts to $K \times \mathbb{R}$, since for any component $S_0 \subset S$ we have $\pi_1 S_0 \subset \ker u = \pi_1 K$. If $-\chi(S) = \|u\|$, then S is incompressible and $\pi_1 S_0 \to \pi_1(K \times \mathbb{R}) = \pi_1(K)$ is injective. Since subgroups of $\pi_1 K$ of infinite index are free, we see that S_0 is a finite cover of K, hence $\|u\| = -\chi(S) \geq -\chi(S_0) \geq -\chi(K)$, as desired. ∎

In fact, we see that any sphereless framed surface S representing u with minimal $-\chi(S)$ is homotopic to the fiber K.

The behaviour of $\| \, \|$ is decisively determined by the fact that integral classes have integral seminorms. We will show:

Theorem 14.5 ([Thu86]) *A seminorm $\| \, \| \colon \mathbb{Z}^n \to \mathbb{Z}$ extends uniquely to a seminorm $\| \, \| \colon \mathbb{R}^n \to [0, \infty)$. A seminorm on $\mathbb{R}^n$ takes integer values on $\mathbb{Z}^n \Leftrightarrow \|x\| = \max\limits_{\ell \in F} |\ell(x)|$, where $F \subset \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$ is finite.*

This enables us to state Thurston's description of the cone $\mathcal{C}$ of nonsingular classes, $\mathcal{C} \subset H^1(M; \mathbb{R}) - \{0\}$.

We will consistently use certain natural isomorphisms of the homology and cohomology groups of M. By the Universal Coefficient Theorem, $H^1(M; \mathbb{Z}) \cong \operatorname{Hom}(H_1(M; \mathbb{Z}); \mathbb{Z})$ and $H_1(M; \mathbb{Z})/\text{torsion} \cong \operatorname{Hom}(H^1(M; \mathbb{Z}); \mathbb{Z})$. With real coefficients, $H^i(M; \mathbb{R})$ and $H_i(M; \mathbb{R})$ are dual vector spaces for any i. By Poincaré Duality, we may identify $H^2(M; \mathbb{Z})$ with $H_1(M; \mathbb{Z})$. Thus we regard the Euler class χ_F of a plane bundle F on M, which is usually taken to be in $H^2(M; \mathbb{Z})$, as an element of $H_1(M; \mathbb{Z})$ and thus as a linear functional on $H^1(M; \mathbb{R})$.

Theorem 14.6 ([Thu86]) *$\mathcal{C}$ is the union of (finitely many) convex open cones in $\operatorname{int}(T_i)$, where T_i is a maximal region on which $\| \, \|$ is linear. The region T_i containing a given nonsingular 1-form ω is $T_i = \{u \in H^1(M; \mathbb{R}) \mid \|u\| = -\chi_F(u)\}$ where χ_F is the Euler class of the plane bundle $F = \ker \omega$.*

Note. When $\| \, \|$ is a norm, we may say that $\mathcal{C}$ is all vectors $v \neq 0$ such that $\frac{v}{\|v\|}$ belongs to certain "nonsingular faces" of the polyhedral unit ball. Incidentally, we have that $\| \, \|$ is a norm $\iff$ all $T^2 \subset M$ separate M $\iff$ all incompressible $T^2 \subset M$ separate M.

We give our own analytic proof of theorem 14.5.

Proof. Clearly $\| \, \|$ extends by homogeneity to a seminorm $\| \, \|$ on $\mathbb{Q}^n$. This function is Lipschitz, hence has a unique continuous extension to a function $\mathbb{R}^n \to [0, \infty[$. The triangle inequality and homogeneity follow by continuity.

By convexity, all one-sided directional derivatives of $N(x) = \|x\|$ exist. Suppose $\tau = (0, \frac{1}{q}p), \quad q \in \mathbb{Z}^+, p = (p_2, \ldots, p_n) \in \mathbb{Z}^{n-1}$ is a rational point. For integral m, we compute

$$\begin{aligned} \frac{\partial_+ N}{\partial x_1}(\tau) &= \lim_{m\to\infty} \frac{N(\tau + 1/qm\, e_1) - N(\tau)}{1/qm} \\ &= \lim_{m\to\infty} (N(1, mp_2, \ldots, mp_n) - N(0, mp_2, \ldots, mp_n)) \\ &\in \mathbb{Z}, \end{aligned}$$

since $\mathbb{Z}$ is closed.

By induction on n, we assume that $N(0, \bar{x})$, $\bar{x} \in \mathbb{R}^{n-1}$, is given by the supremum of finitely many functionals $\ell(\bar{x}) = a_2x_2 + \cdots + a_nx_n, \quad a_2, \ldots, a_n \in \mathbb{Z}$, $\bar{x} = (x_2, \ldots, x_n)$. By convexity, any supporting line L to graph$(N) \subset \mathbb{R}^n \times \mathbb{R}$ lies in a supporting hyperplane H (**supporting** means intersects the graph without passing above it.) We choose $\bar{x}$ a rational point for which $N|0 \times \mathbb{R}^{n-1}$ is locally given by ℓ and choose L to pass through $(0, \bar{x}, N(0, \bar{x})) \in \mathbb{R}^n \times \mathbb{R}$ in the direction of $(1, 0, \frac{\partial_+ N}{\partial x}(0, \bar{x}))$. Then we see that H is uniquely determined as the graph of $(\frac{\partial_+ N}{\partial x_1}(0, \bar{x}))x_1 + a_2x_2 + \cdots + a_nx_n$. So for a dense set of $\bar{x}$, the graph of N has a supporting functional at $(0, \bar{x})$ with integral coefficients. supporting

Reasoning for each integrally defined hyperplane as we have for $\{x_1 = 0\}$, we find integral supporting functionals $\ell(x) = a_1x_1 + \cdots + a_nx_n, \quad a_i \in \mathbb{Z}$, to the graph of N exist at a dense set in $\mathbb{R}^n$. Since N is Lipschitz, there is a bound $|a_i| \leq K, \quad i = 1, \ldots, n$. Thus the supporting functionals form a finite set F, so $S(x) = \sup\limits_{\ell \in F} |\ell(x)|$ is clearly a seminorm. But $S(x) \leq N(x)$ and equality holds on a dense set, implying that $S(x) = N(x)$ by continuity. ∎

Before giving the proof of theorem 14.6, let us observe one elementary consequence of theorem 14.5. Since $\|\ \|$ is natural, any diffeomorphism $h\colon M \to M$ induces an isometry h^* of $H^1(M; \mathbb{R})$. If $\|\ \|$ is a norm, then the finite set of vertices of the unit ball spans $H^1(M; \mathbb{R})$ and is permuted by h^*.

Corollary 14.7 *If all incompressible $T^2 \subset M$ separate M, then the image of* Diff(M) *in $GL(H^1(M; \mathbb{R}))$ is finite.*

Proof. Suppose ω, ω' are nonsingular closed 1-forms that are C^0 close. Then the oriented plane fields $F = \ker\omega$, $F' = \ker\omega'$ are homotopic and so determine the same Euler class $\chi_{F'} = \chi_F \in H_1(M; \mathbb{R})$.

If $[\omega']$ is rational, let $q[\omega'] = \beta' \in H^1(M;\mathbb{Z})$, where $0 < q \in \mathbb{Q}$ and β' is indivisible. Then if K' is the (connected) fiber of the fibration associated to $q\omega'$, we have $\chi(K') = \chi_{F'}(K') = \chi_F(K')$. Using this and proposition 14.4, we find $\|[\omega']\| = \frac{1}{q}(-\chi(K')) = -\frac{1}{q}\chi_F(K') = \chi_F[\omega']$. Thus for all rational classes $[\omega']$ near $[\omega]$, $\| \, \|$ is given by the linear functional $-\chi_F$. This show that $\| \, \|$ agrees with $-\chi_F$ on a neighborhood of any nonsingular class $[\omega]$, as desired.

It only remains to show that every $\alpha \in \operatorname{int}(T)$ is a nonsingular class, where $T = \{\alpha \in H^1(M;\mathbb{R}) \mid \|\alpha\| = -\chi_F(\alpha)\}$ is the largest region containing $[\omega]$ on which $\| \, \|$ is linear.

For this, we need a result of Thurston's thesis [Thu72] concerning the isotopy of an incompressible surface $S \subset M$ when M is foliated without "dead end components". In fact, this result is only explicitly stated for tori, and one must see [Rou73] for a published account of this case. Restricting our attention to the foliation $\mathcal{F}$ defined by ω ($\mathcal{F}$ is tangent to $\ker\omega = F$), we may state this result as follows: any incompressible, oriented and connected surface $S_0 \subset M$ with $-\chi(S_0) \geq 0$ may be isotoped so as to either lie in a leaf of $\mathcal{F}$ or so as to have only saddle tangencies with $\mathcal{F}$. (We call a tangency point s of
saddle S_0 with $\mathcal{F}$ a **saddle** if for some open ball B around s, the map $\int_s^x \omega \colon B \cap S_0 \to \mathbb{R}$ has a non-degenerate critical point at s which is not a local extremum.)

Suppose $\alpha \in T \cap H^1(M;\mathbb{Z})$ is not a multiple of $[\omega]$. Represent α by a framed sphereless surface with $-\chi(S) = \|\alpha\|$. As S is incompressible, each component of S may be isotoped (independently) to a surface S_i which either lies in a leaf of $\mathcal{F}$ or has only saddle tangencies with $\mathcal{F}$. If some S_i lies in a leaf L of $\mathcal{F}$, then (as in proposition 14.4) $\pi_1 S_i$ would be of finite index in $\pi_1 L = \ker[\omega]$. Since $\pi_1 S_i \subset \ker\alpha$, we would find that α is a multiple of $[\omega]$. Thus each S_i has only saddle tangencies with $\mathcal{F}$.

Lemma 14.8 *For each i, the normal orientations of S_i and $\mathcal{F}$ agree at all tangencies.*

Proof. We compute $\|\alpha\|$ in two ways. First, $\|\alpha\| = -\chi(S) = \sum_i -\chi(S_i)$ Choosing some Riemannian metric on M, we may use the vector field V_i on S_i dual to $\omega|S_i$ to compute $-\chi(S_i)$. V_i will have only non-degenerate zeroes of index -1, since all tangencies are saddles. The Hopf Index theorem [Mil66] gives $-\chi(S_i) = n_i$, where n_i is the number of tangencies of S_i with $\mathcal{F}$. Thus $\|\alpha\| = \sum n_i$.

On the other hand, we know that $\alpha \in T$ implies $\|\alpha\| = -\chi_F(\alpha)$. The natural normal orientations of F and S gives us preferred orientations on F and S_i, for each i. Each oriented plane bundle $F|S_i$ has an Euler class $\chi_F(S_i)[S_i]$ where $[S_i] \in H^2(S_i;\mathbb{Z})$ is the orientation class. We compute $\chi_F(S_i)$ as the self-intersection number of the zero section of $F|S_i$. For the purpose, look at the field W_i of vectors on S_i tangent to $\mathcal{F}$, which are the projection onto F of the unit normal vectors of S_i. Regarding W_i as a perturbation of the zero section of $F|S_i$, we compute the self-intersection number using the local orientations of F and S_i. When these orientations agree, one counts the singularity as -1 (just as in the tangent bundle case already considered) but when the orientations disagree one counts $+1$. Thus $-\chi_F(S_i) = n_i^+ - n_i^-$, where n_i^+ is the number of tangencies at which the orientations agree and n_i^- is the number of tangencies at which the orientations disagree. Thus $\|\alpha\| = \sum n_i^+ - n_i^-$.

Since $n_i = n_i^+ + n_i^-$, we have $\sum n_i^+ + \sum n_i^- = \|\alpha\| = \sum n_i^+ - \sum n_i^-$, whence all the nonnegative integers n_i^- must be zero. This proves the lemma. ∎

Because of the lemma, we may define a framing N_i of S_i with $\omega(N_i) > 0$ everywhere. This framing may be extended to a product neighborhood structure on $U_i \supset S_i$, where $h\colon S_i \times [-1,1] \to U_i$ is a diffeomorphism, $h_*(\frac{\partial}{\partial t}) = N_i$ on $S_i = S_i \times 0$ and $\omega(h_*(\frac{\partial}{\partial t})) > 0$. Let $B\colon [-1,1] \to [0,\infty]$ be a smooth function vanishing on $|x| > \frac{1}{2}$ with $\int_{-1}^{+1} B = +1$. Letting $\eta_i = (\pi_2 h^{-1})^* B dt$ we find that, for all $s > 0$, $(\omega + s\eta_i)(h_* \frac{\partial}{\partial t}) > 0$ on U. But since $\omega + s\eta_i = \omega$ away from U, we see that the closed 1-form $\omega + s\eta_i$ is nonsingular.

The portion of theorem 14.6 already proven gives $[\omega + s\eta_i] \in \operatorname{int} T$. Thus, $[\eta_i] = \lim_{s\to\infty} \frac{[\omega + s\eta_i]}{s} \in T \cap H^1(M;\mathbb{Z})$, for all i. So replacing $[\omega]$ by $[\omega] + s_1[\eta_1] + \cdots + s_{i-1}[\eta_{i-1}]$, we see inductively that $[\omega] + s_1[\eta_1] + \cdots + s_i[\eta_i]$ is nonsingular for all $s_1, \ldots, s_i \geq 0$. In particular, for all $s \geq 0$, $[\omega] + s\alpha = [\omega] + s\sum[\eta_i]$ is nonsingular.

We just showed that if $\beta = [\omega] \in \operatorname{int} T$ is a nonsingular class, then $\beta + s\alpha$ is nonsingular for all $\alpha \in T \cap H^1(M;\mathbb{Z})$ and $s \geq 0$. Now consider an arbitrary $\gamma \in \operatorname{int} T$, $\gamma \neq \beta$. By convexity we may find $v_1, \ldots, v_d \in \operatorname{int} T$, $d = \dim H^1(M;\mathbb{R})$, so that γ is in the interior of the d-simplex spanned by $\beta, v_1, \ldots, v_d$. We may choose $v_1, \ldots, v_d$ rational, say $v_j = \frac{1}{N}\alpha_j$, some $N \in \mathbb{Z}^+$, $\alpha_j \in \operatorname{int} T \cap H^1(M;\mathbb{Z})$. We have $\gamma = t_0\beta + \sum_{j=1}^d t_j\alpha_j$, with all $t_j > 0$. By induction on k, we see that each $\beta + \sum_{j=1}^k (t_j/t_0)\alpha_j$ is nonsingular. Setting $k = d$ and multiplying by

$t_0 > 0$, we see that γ is nonsingular as well. Thus if one point $\beta \in \operatorname{int} T$ is nonsingular, all $\gamma \in \operatorname{int} T$ are nonsingular. ∎

atoroidal

We will sharpen Thurston's theorem 14.6 in the case when m is **atoroidal** (contains no incompressible imbedded tori) and $H^1(M;\mathbb{Z}) \ncong \mathbb{Z}$. We show (theorem 14.11) that a nonsingular face T (i.e., one containing a nonsingular class) of the unit $\| \|$-ball determines a canonical flow $\phi_t \colon M \to M$ such that $\operatorname{int} T$ consists precisely of all $[\omega]$ where ω is a closed 1-form with $\omega(\frac{\partial \phi}{\partial t}) > 0$. We must begin by relating the atoroidal condition to Thurston's classification of surface homeomorphisms.

We suppose $f \colon M \to S^1$ is a fibration. Then flows ψ_t for which $\frac{d}{dt} f(\psi_t m) > 0$ (we will only consider flows having a continuous time derivative) determine an isotopy class of surface homeomorphisms. For any $k \in K = f^{-1}(1)$, we consider the smallest time $T(k) > 0$ for which $\psi_{T(k)}(k) \in K$. This map $T(k) \colon K \to (0, \infty)$ is smooth (since the flow lines of ψ are transverse to K) and the **return map** $R(k) = \psi_{T(k)}(k)$ is a homeomorphism. By varying ψ, we obtain an isotopy class of homeomorphisms of the fiber K as return maps; this *isotopy* class will be called the **monodromy** of f and denoted $m(f)$.

return map

monodromy

We remark that the monodromy of f is determined algebraically by the cohomology class $\beta = f^*[d\theta] \in H^1(M;\mathbb{Z})$, or equivalently by the map $f_* \colon \pi_1 M \to \pi_1 S^1$. First assume that β is indivisible. From the exact homotopy sequence $1 \longrightarrow \pi_1 K \longrightarrow \pi_1 M \xrightarrow{f_*} \pi_1 S^1 \longrightarrow 1$, we see that $\pi_1 M$ is the semidirect product $\pi_1 K \ltimes_\alpha \mathbb{Z}$, where α is the outer automorphism of $\pi_1 K$ determined by the monodromy of f. Thus $\pi_1 K$ ($= \ker f_*$) and α are determined by f_* alone. Clearly the topological type of K is determined by $\pi_1 K$; but Nielsen also showed that isotopy classes in Diff(K) correspond 1-1 to outer automorphisms of $\pi_1 K$. In general, $\beta = n\beta'$ is a positive integer multiple of an indivisible class β', and n is determined by $\operatorname{coker} f_* = \mathbb{Z}/n\mathbb{Z}$. We see that the fiber of f consists of n copies of K (where $\pi_1 K = \ker f_*$) which are permuted cyclically by the monodromy. The nth power of the monodromy preserves K and acts on $\pi_1 K$ by α (the outer automorphism of $\ker f_*$.) Thus we may unambiguously speak of the monodromy of a nonsingular class $\beta \in H^1(M;\mathbb{Z})$.

pseudo-Anosov

strict conjugacy

We say that the monodromy $m(f)$ of a fibration $f \colon M \to S^1$ is **pseudo-Anosov** if the isotopy class has a pseudo-Anosov representative R. This representative is then uniquely determined within **strict conjugacy**, that is for any two pseudo-Anosov representatives $R_0, R_1 \in m(f)$ there will be a homeomorphism g isotopic to the identity for which $R_0 g = g R_1$.

Proposition 14.9 *Suppose that $H^1(M;\mathbb{Z}) \neq \mathbb{Z}$. Given a fibration $f\colon M \to S^1$, M is atoroidal precisely when the monodromy $m(f)$ is pseudo-Anosov and the fibers of f are not composed of tori.*

Proof. Suppose M contains an incompressible torus S and let $\mathcal{F}$ be the foliation of M by the fibers of f. Again using the result of Thurston's thesis discussed in the proof of theorem 14.6 [Rou73, Thu72], we may isotope S to either lie in a leaf of $\mathcal{F}$ or to be transverse to $\mathcal{F}$ (since $\chi(S) = 0$, the presence of saddle tangencies would force there to be tangencies of other types.) If S does lie in a leaf, then the fibers of f are composed of tori parallel to S. If the torus S is transverse to $\mathcal{F}$, then one may define a flow ψ on M that preserves S and satisfies $\frac{d}{dt}(f \circ \psi_t) = 1$. Thus the return map $\psi_1\colon K \to K$, $K = f^{-1}(1)$, preserves the family of curves $S \cap K$. Since S is incompressible, each of those curves is homotopically nontrivial in K. If the monodromy of f were pseudo-Anosov, these curves would grow exponentially in length under iteration by ψ_1. So we see that when $m(f)$ is pseudo-Anosov and the fibers of f are not unions of tori, then M must be atoroidal.

Conversely, when the fibers of f are unions of tori, these tori are essential. So we assume the components of the fibers have higher genus and that the monodromy is not pseudo-Anosov (hence reducible or periodic) and look for an incompressible torus. If $m(f)$ is reducible, we may construct ψ with $\frac{d}{dt}(f \circ \psi_t) = 1$ for which ψ_1 cyclically permutes a family of homotopically nontrivial closed curves $C \subset K$. Then $\{\psi_t C\}$ is an incompressible torus. If $m(f)$ has period n, after Nielsen (see exposé 11), we may choose ψ with $\frac{d}{dt}(f \circ \psi_t) = 1$ for which $\psi_n =$ identity. Thus M is Seifert fibered. One may easily compute that $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g+1}$, where g is the genus of the topological surface which is the orbit space of ψ [Orl72]. As we assumed $H^1(M;\mathbb{Z}) \neq \mathbb{Z}$, we must have a homologically nontrivial curve in this orbit space which corresponds to an incompressible torus in M. ∎

We may consider flows transverse to a fibration over S^1 from three viewpoints. The first is to begin with the fibration and produce transverse flows suspension

cross-section

and an isotopy class of return maps. The second is to begin with a homeomorphism $R\colon K \to K$ and produce a fibration over S^1 with fiber K and a transverse flow ϕ with return map R. This is the well-known mapping torus construction, for which one sets $X = K \times [0,1]/(k,1) = (R(k),0)$, $f\colon X \to ([0,1]/0=1) = S^1$ the natural fibration and defines ψ to be the flow along the curves $k \times [0,1]$ with unit speed. Clearly $\psi_1|K \times 0 = R$ is the return map of ψ, as desired. This flow ψ is called the **suspension** of R. The third viewpoint is to begin with a flow ψ on X and to seek a fibration f over S^1 to which ψ is transverse — a fiber K is called a **cross-section** to ψ. Note that K and ψ determine the return map R and an isotopy class of fibrations f.

In general, one has little hope of finding cross-sections, since many manifolds don't fiber over S^1 at all. But there is a classification of the fibrations transverse to ψ which is especially concrete in the case of interest to us now.

loop

minimal

Suppose that some cross-section K to a flow ϕ has a return map $R\colon K \to K$ admitting a Markov partition $\mathcal{M} = \{S_1, \dots, S_m\}$ (see exposé 10 — the case we need is when R is pseudo-Anosov). There is a directed graph with vertices $S_1, \dots, S_m$ and arrows $S_i \to S_j$ for each i and j for which $R(S_i)$ meets $\mathrm{int}(S_j)$. A **loop** ℓ for $\mathcal{M}$ is a cyclic sequence of arrows $S_{i_1} \to S_{i_2} \to \cdots \to S_{i_k} \to S_{i_1}$. Each loop ℓ determines a periodic orbit for R and thus a periodic orbit $\gamma(\ell)$ for ϕ. If all of $i_1, \dots, i_k$ are distinct, we call ℓ **minimal**. There are only finitely many minimal loops ℓ.

homology direction

We now discuss the classification and existence of cross-sections to flows. Given a flow ψ on a compact manifold X there is a nonempty compact set of homology directions $D_\psi \subset H_1(X;\mathbb{R})/\mathbb{R}^+$, where the quotient space is topologized as the disjoint union of the origin and unit sphere. A **homology direction** for ψ is an accumulation point of the classes determined by long, nearly closed trajectories of ψ. We note that when K is a cross-section to ψ, K is normally oriented by ψ and so determines a dual class $u \in H^1(X;\mathbb{Z})$. Let $\mathcal{C}_{\mathbb{Z}}(\psi) := \{u \in H^1(X;\mathbb{Z}) \mid u \text{ is dual to some cross-section } K \text{ to } \psi\}$.

Theorem 14.10 ([Fri82b, Fri76]) $\mathcal{C}_{\mathbb{Z}} = \{u \mid u(D_\psi) > 0\}$. *If ϕ, as above, has a cross-section K and the return map R admits a Markov partition $\mathcal{M}$, then $\mathcal{C}_{\mathbb{Z}}(\phi) = \{u \mid u(\gamma(\ell)) > 0$ for all minimal loops ℓ for $\mathcal{M}\}$.*

Thus $\mathcal{C}_{\mathbb{Z}}(\psi)$ consists of all lattice points in a (possibly empty) open convex cone $\mathcal{C}_{\mathbb{R}}(\psi) := \{u \mid u(D_\psi) > 0\} \subset H^1(X;\mathbb{R}) - \{0\}$. It follows easily from theorem 14.10 that $\mathcal{C}_{\mathbb{R}}(\psi) = \{[\omega] \mid \omega \text{ is a closed 1-form with } \omega(\frac{d\psi}{dt}) > 0\}$.

Returning to our discussion of three-manifolds, we call a flow ψ on M pseudo-Anosov if it admits some cross-section for which the return map is pseudo-Anosov. We now describe the cross-sections to pseudo-Anosov flows, and show they are uniquely determined by their homotopy class among non-singular flows on M.

Theorem 14.11 *Suppose M fibers over S^1. Then each flow ψ on M that admits a cross-section determines a nonsingular face $T(\psi)$ for the norm $\| \, \|$ on $H^1(M;\mathbb{R})$. Here $T(\psi) = \{\|u\| = -\chi_{\psi^\perp}(u)\}$ and $\psi^\perp$ denotes the normal plane bundle to the vector field $\frac{d\psi}{dt}$. One has $\mathcal{C}_{\mathbb{R}}(\psi) \subset \operatorname{int} T(\psi)$.*

For any pseudo-Anosov flow ϕ on M, $\mathcal{C}_{\mathbb{R}}(\phi) = \operatorname{int} T(\phi)$.

The face $T(\phi)$ (or the class $\chi_{\phi^\perp}$) determines the pseudo-Anosov flow ϕ up to strict conjugacy. Thus any nonsingular face T on an atoroidal M with $H^1(M;\mathbb{Z}) \neq \mathbb{Z}$ determines a strict conjugacy class of pseudo-Anosov flows.

Proof. For $u \in \mathcal{C}_{\mathbb{Z}}(\psi)$, there is a cross-section K to ψ dual to u. We have $\|u\| = -\chi(K)$, by proposition 14.4. Since the restriction $\psi^\perp|K$ is the tangent bundle of K, we have $-\chi(K) = -\chi_{\psi^\perp}(u)$. Thus $-\chi_{\psi^\perp}$ is a linear functional on $H^1(M;\mathbb{R})$ that agrees with $\| \, \|$ on $\mathcal{C}_{\mathbb{Z}}(\psi)$ and the first paragraph of theorem 14.11 is shown.

We now observe

Lemma 14.12 *Any cross-section K to a pseudo-Anosov flow ϕ on M will have pseudo-Anosov return map R_K.*

Proof. By definition, there is some cross-section L to ϕ with pseudo-Anosov return map R_L, but K and L will generally not be homeomorphic (one calls return maps to distinct cross-sections **flow-equivalent**.) In any case, any structure on L invariant under R_L is carried over to a structure on K invariant under R_K under the system of local homeomorphisms between K and L determined by ϕ. This shows that R_K preserves a pair of transverse foliations $\mathcal{F}^u_K$ and $\mathcal{F}^s_K$ with the same local singularity structure as a pseudo-Anosov diffeomorphism. flow-equivalent

We now show that the closure $\overline{P}$ of any prong P of $\mathcal{F}^u_K$ or $\mathcal{F}^s_K$ is the component K_0 of K which contains P. By passing to a cyclic cover $M_n \to M$ determined by the composite homeomorphism $\pi_1 M \to (\pi_1 M/\pi_1 K_0) \cong \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

and restricting to the cross-section $K_0 \subset M_n$ we may assume that K is connected and that R_K leaves P invariant (choose n so that P is invariant under $R^n_{K_0}$). Consider the closed R_L invariant subset $\{\phi_t\overline{P}\} \cap L = I$. Since I contains the closure of a prong for the pseudo-Anosov diffeomorphism R_L, we know that I is dense in some component $L_0 \subset L$. As L_0 is a cross-section to ϕ, we find that $\{\phi_t\overline{P}\} = M$. As $\overline{P}$ is R_K invariant, we find $\overline{P} = K$ as desired.

Similarly we can check that the foliations $\mathcal{F}^u_K$ and $\mathcal{F}^s_K$ have no closed leaves.

It follows by the Poincaré-Bendixson theorem that each leaf closure contains a singularity, and thus a prong. So we find that all leaves of $\mathcal{F}^s_K$ and $\mathcal{F}^u_K$ are dense in their component of K.

We may see from this density of leaves and the fact that the local stretching and shrinking properties of R_K are the same as those of R_L that the Markov partition construction of exposé 10 works for R_K. (It is easiest to construct birectangles for R_K by "analytic continuation" from immersed birectangles in L. This makes sense because K and L have the same universal cover.) As in the Anosov case [RS75], the Parry measures for the one-sided subshifts of finite type associated to $\mathcal{M}$ push forward to give transverse measures on $\mathcal{F}^u_K$ and $\mathcal{F}^s_K$ that transform under R_K by factors λ_K^{-1} and λ_K, for some $\lambda_K > 1$. As leaves are dense, these measures have positive values on any transverse interval but vanish on points. Thus R_K is pseudo-Anosov. ∎

Now suppose that ϕ^1 and ϕ^2 are pseudo-Anosov flows on M for which $\mathcal{C}_{\mathbb{R}}(\phi^1)$ intersects $\mathcal{C}_{\mathbb{R}}(\phi^2)$. Then we may choose $u \in \mathcal{C}_{\mathbb{R}}(\phi^1) \cap \mathcal{C}_{\mathbb{R}}(\phi^2) \cap H^1(M;\mathbb{Z})$ and find fibrations $f_i \colon M \to S^1$ with $\frac{d}{dt}(f_i \circ \phi^i_t) > 0$ and $u = [f_i^*(d\theta)]$, $i = 1, 2$.

As discussed earlier, u determines $m(f_i)$. This gives a homeomorphism $h \colon M \to M$ such that $f_1 \circ h = f_2$ where h acts on $\pi_1 M$ by the identity. Thus h is isotopic to the identity [Wal68]. Hence, by this preliminary isotopy, we assume $f_1 = f_2 = f$ and denote the fiber by K.

Each ϕ^i determines a return map $R_i \colon K \to K$. By the lemma above, these R_i are pseudo-Anosov. Since the maps R_i are in the same isotopy class $h(f)$, they are strictly conjugate by the uniqueness of pseudo-Anosov diffeomorphisms (exposé 12).

Now suppose that $gR_1 = R_2 g$, with g isotopic to the identity. Then the map $C_0 \colon M \to M$ defined by $C_0(\phi^1_s k) = (\phi^2_s g k)$, $k \in K$, $0 \le s \le 1$, is a homeomorphism conjugating flows ϕ^1 and ϕ^2 and $f \circ C_0 = f$. As $C_0|K = g$ is isotopic to the identity, C_0 may be isotoped to C_1 where $f \circ C_t = f$, for $t \in [0,1]$ and C_1 fixes K. Since Diff K is simply connected [Ham66], we may isotope C_1 to the identity C_2 (through C_t satisfying $f \circ C_t = f$, $t \in [1,2]$.)

We have shown so far that if ϕ^i are pseudo-Anosov flows, $i = 1,2$, then either $\mathcal{C}_{\mathbb{Z}}(\phi^1)$ equals $\mathcal{C}_{\mathbb{Z}}(\phi^2)$ or is disjoint from it, since conjugating a flow by conjugacy isotopic to the identity doesn't affect $\mathcal{C}_{\mathbb{Z}}$. It follows easily that the open cones $\mathcal{C}_{\mathbb{R}}(\phi^1)$ and $\mathcal{C}_{\mathbb{R}}(\phi^2)$ are either disjoint or equal.

Now suppose that ϕ is pseudo-Anosov but $\mathcal{C}_{\mathbb{R}}(\phi)$ is a proper subcone of $\operatorname{int} T(\phi)$. By theorem 14.10, $\mathcal{C}_{\mathbb{R}}(\phi)$ is defined by linear inequalities with integer coefficients, and so there is an integral class $u \in \operatorname{int} T \cap \partial\mathcal{C}_{\mathbb{R}}(\phi)$. Then u is nonsingular (theorem 14.6), the fibration corresponding to u has pseudo-Anosov monodromy (proposition 14.9) and one obtains an Anosov flow ψ with $u \in \mathcal{C}_{\mathbb{R}}(\psi)$. This shows that $\mathcal{C}_{\mathbb{R}}(\psi)$ and $\mathcal{C}_{\mathbb{R}}(\phi)$ are neither disjoint nor equal, contradicting the previous paragraph.

Thus we see that pseudo-Anosov flows satisfy $\mathcal{C}(\phi) = \operatorname{int} T(\phi)$. ∎

Theorem 14.11 shows that pseudo-Anosov maps satisfy an interesting extremal property within their isotopy class. Suppose $h_0 \colon K \to K$ has suspension flow $\psi_t^0 \colon M \to M$, where we take K connected and dual to the indivisible class $u \in H^1(M;\mathbb{Z})$. Given an isotopy h_t starting at h_0, we may deform ψ^0 through flows ψ^t with cross-section K and return map h_t. We regard $u^{-1}(1)$ as a subset of $H_1(M;\mathbb{R})/\mathbb{R}^+$ and note that we always have $D_{\psi^t} \subset u^{-1}(1)$. By the Wang exact sequence:

$$H_1(K;\mathbb{R}) \xrightarrow{h_{0*}-\mathrm{Id}} H_1(K;\mathbb{R}) \longrightarrow H_1(M;\mathbb{R}) \xrightarrow{u} \mathbb{R} \longrightarrow 0,$$

we may identify $u^{-1}(1)$ with $u^{-1}(0) = \operatorname{coker}(h_{0*} - \mathrm{Id})$ by some fixed splitting of u. Whenever $h_s = h_t$, the simple connectivity of Diff K [Ham66] implies that $D_{\psi^s} = D_{\psi^t}$. Thus we may unambiguously associate a set of homology directions $D_h \subset \operatorname{coker}(h_{0*} - \mathrm{Id})$ to homeomorphisms h isotopic to h_0. Now assume h_0 is pseudo-Anosov. By theorem 14.11, we have $\mathcal{C}_{\mathbb{R}}(\psi^s) \subset \operatorname{int} T(\psi^s) = \operatorname{int} T(\psi^0) = \mathcal{C}_{\mathbb{R}}(\psi^0)$. Thus we find, using theorem 14.10, that the convex hull of D_{h_s} (which may be identified with the asymptotic cycles of ψ^s in this situation [Fri82a, Sch57]) always contains the convex polygon determined at $s = 0$. Thus we may say that pseudo-Anosov diffeomorphisms have the fewest generalized rotation numbers in their isotopy class.

We may analyze the topological entropy of the return-maps R_K of the various cross-sections K to a pseudo-Anosov flow $\mathcal{C}$. We parameterize these cross-sections K by their dual classes $u \in H^1(M;\mathbb{Z})$ and define $h\colon \mathcal{C}_{\mathbb{Z}}(\phi) \to (0,\infty)$ by $h([K]) = h(R_K)$, the topological entropy of R_K. We showed in [Fri82a] that $1/h$ extends uniquely to a homogeneous, downwards convex function $1/h\colon \overline{\mathcal{C}_{\mathbb{R}}(\phi)} \to [0,\infty]$ that vanishes exactly on $\partial\mathcal{C}_{\mathbb{R}}(\phi)$. Thus $h(u)$ may be defined for all $u \in H^1(M;\mathbb{R})$ in a natural way. The smallest value of h on $\operatorname{int} T \cap \{\|u\| = 1\}$ defines an interesting measure of the complexity of ϕ (or equivalently, by theorem 14.11, of the face $T = T(\phi)$). The integral points at which h is the largest give the "simplest" cross-section to the flow ϕ (see [Fri82a]).

If one is given a pseudo-Anosov diffeomorphism $h\colon K \to K$ and a Markov partition $\mathcal{M}$ for h, theorems 14.10 and 14.11 give an effective description of the nonsingular face T determined by the suspended flow $\phi_t\colon M \to M$ of h, in terms of the orbits corresponding to minimal loops. As the computation of minimal loops in a large graph is difficult, we observe that there is a more algebraic way of using $\mathcal{M}$ to obtain a system of inequalities defining T. (We refer the reader to [Fri82a] for details, where we used this method to construct a rational zeta function for axiom A and pseudo-Anosov flows.) For sufficiently fine $\mathcal{M}$, we may associate to $\mathcal{M}$ a matrix A with entries in $H_1(M;\mathbb{Z})/\text{torsion} = H$. The expression $\det(I - A)$, regarded as an element in the group ring of the free abelian group H, may be uniquely written as $1 + \sum a_i g_i$, $g_i \in H - \{0\}$, $a_i \in \mathbb{Z} - \{0\}$, g_i distinct. Then T is defined by the inequalities $u(g_i) > 0$.

To illustrate Thurston's theory, it is convenient to work on a bounded M^3. The norm considered above can be extended to such M by omitting spheres and discs before computing the negative Euler characteristic. One should restrict to the case where ∂M is incompressible, and then theorems 14.2 and 14.6 and proposition 14.4 extend [Hem76, Thu86].

We let K be the quadruply connected planar region and h the indicated composite of the two elementary braids (figure 14.1) which fixes the outer boundary component. We will let M be the mapping torus of h and compute $\|\ \|$. Rather than finding a pseudo-Anosov map isotopic to h, which would only help compute one face, we will instead compute $\ker(u\colon \pi_1 M \to \mathbb{Z})$ for several indivisible $u \in H^1(M;\mathbb{Z})$. When this kernel is finitely generated, the-

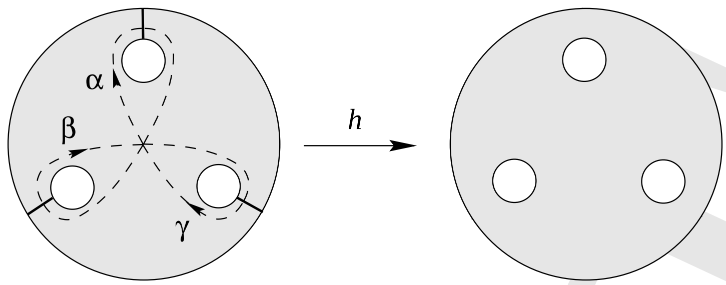


Figure 14.1:

orem 14.2 shows u is nonsingular and proposition 14.4 enables us to compute $\|u\|$. From a small collection of values of $\| \, \|$, theorem 14.6 allows us to deduce all the others, indicating the existence of nonsingular classes that would be hard to detect using only theorem 14.2.

We first compute $\pi_1 M = \pi_1 K \ltimes \mathbb{Z}$. Writing $\pi_1 K$ as the free group on the loops α, β and γ shown in the diagram, we find:

$$\begin{aligned}
\pi_1 M &= \left\langle \alpha, \beta, \gamma, t \;\middle|\; t^{-1}\alpha t = \gamma,\, t^{-1}\beta t = \gamma^{-1}\alpha\gamma,\, t^{-1}\gamma t = (\gamma^{-1}\alpha\gamma)\beta(\gamma^{-1}\alpha\gamma)^{-1} \right\rangle \\
&= \left\langle \alpha, \beta, \gamma, t \;\middle|\; t^{-1}\alpha t = \gamma,\, t^{-1}\beta t = \gamma^{-1}\alpha\gamma,\, \gamma\beta t = \beta t\beta \right\rangle \\
&= \left\langle \gamma, t \;\middle|\; (t\gamma^{-1}t\gamma t^{-1}\gamma)^2 = \gamma(t\gamma^{-1}t\gamma t^{-1}\gamma)t \right\rangle .
\end{aligned}$$

Abelianizing gives $H_1(M;\mathbb{Z}) = \mathbb{Z}\gamma \oplus \mathbb{Z}t$. Suppose $u \in H^1(M;\mathbb{Z})$ is indivisible, so that $a = u(\gamma)$ and $b = u(t)$ are relatively prime. The Reidemeister-Schreier process gives a presentation for $\ker(u\colon \pi_1 M \to \mathbb{Z})$ (essentially by computing the fundamental group of the infinite cyclic covering corresponding to u) which is very ungainly for large a. When $a = 1$, one finds the relatively simple expression:

$$\ker u = \left\langle t_i \;\middle|\; t_i\, t_{i+b-1}\, t_{i+b}^{-1}\, t_{i+b+1}\, t_{i+2b}\, t_{i+2b+1}^{-1} = t_{i+1}\, t_{i+b}\, t_{i+b+1}^{-1}\, t_{i+b+2} \right\rangle .$$

For $b > 1$, this relation expresses t_i in terms of $t_{i+1}, \ldots, t_{i+2b+1}$ and expresses t_{i+2b+1} in terms of $t_i, \ldots, t_{i+2b}$. Thus $\ker u$ is free on $t_1, \ldots, t_{2b+1}$. Similarly, if $b < -1$, then $\ker u$ is free on $t_1, \ldots, t_{1-2b}$ and if $b = 0$, then $\ker u$ is free on t_1, t_2, t_3. If $b = \pm 1$, however, one may abelianize and obtain $(\ker u)^{\mathrm{ab}} = \mathbb{Z}[t, t^{-1}]/(2t^3 - 3t^2 + 3t - 2)$ which maps onto the collection of all 2^nth roots of unity, and so $\ker u$ is not finitely generated.

By theorem 14.2 (Stallings), there is a fibration for $u = (1, b)$ when $b \neq \pm 1$, with fiber K_u satisfying $\pi_1(K_u) = \ker u$. By proposition 14.4, $\|u\| = -\chi(K_u)$, which is clearly

$$-1 + \operatorname{rank}(H_1(K_u)) = \begin{cases} |2b| & b > 1,\, b \in \mathbb{Z} \\ 2 & b = 0 \end{cases} .$$

We will see that these values determine $\| \ \|$ completely. Using the dual basis to (γ, t), we know that:

$$\|(1, b)\| = \begin{cases} |2b| & b > 1,\, b \in \mathbb{Z} \\ 2 & b = 0 \end{cases} .$$

But $\|(1, b)\|$ is a convex function f of b by theorem 14.3 and it takes integer values at integer points. By convexity, $f(1)$ must be 2 or 3. Were $f(1) = 3$, convexity would force

$$f(x) = \begin{cases} 2 + x & \text{for } 0 \leq x \leq 2 \\ 2x & \text{for } x \geq 2 \end{cases}$$

and then $(1, 2)$ would not lie in an open face of the unit ball, contradicting theorem 14.6. Thus one must have $f(1) = 2$, and likewise, $f(-1) = 2$. By convexity, we find $f(x) = \max(|2x|, 2)$. Homogenizing shows $\|(a, b)\| = \max(|2a|, |2b|)$, i.e., $\|u\| = \max(|u(2\gamma)|, |u(2t)|)$.

By theorem 14.6, $u \in H^1(M; \mathbb{R})$ is nonsingular $\iff |u(\gamma)| \neq |u(t)|$.

This example embeds in a larger one, constructed with the mapping torus M_0 of the transformation h^3 (M_0 is a triple cyclic cover of M). $H_1(M_0; \mathbb{Z})$ is free abelian on α, β, γ, t, so there is a norm on $H^1(M_0; \mathbb{R})$ whose restriction to $H^1(M; \mathbb{R}) \cong \{u \in H^1(M_0; \mathbb{R}) \mid u(\alpha) = u(\beta) = u(\gamma)\}$ is $3\| \ \|$. We leave its computation as an exercise.

Chapter 15

A presentation of the group of diffeotopies of a compact, orientable surface.

by F. Laudenbach

Appendix A

The “pair of pants” decomposition of a surface.

by A. Fathi

In the first section, we will give a proof of the inequality used in the proof of theorem 4.4. In the second part, we apply this inequality to modify by a twist a pair-of-pants decomposition of the surface M.

First part:

We have a system $\alpha_0, \alpha_1, \ldots, \alpha_k$ of simple, mutually disjoint curves on M. On the other hand, γ is a simple curve whose intersection with each α_j is minimal (among curves isotopic to γ). We are given positive integers n_j. We construct Γ by making a *positive* twist along the α_j operate on γ, for $j = 0, \ldots, k$. (The notion of positive twist does not depend on the orientation of M.)

Proposition A.1 *For each simple curve β, we have the formula*

$$\left| i([\Gamma], [\beta]) - \sum_j n_j \cdot i([\gamma], [\alpha_j]) \cdot i([\alpha_j], [\beta]) \right| \leq i([\gamma], [\beta])$$

where “[ ]” means “isotopy class of”.

Proof. For such a curve Γ as described, Γ coincides with γ outside of tubular neighbourhoods of α_j. The position of Γ and γ with the [???] endpoints of a common arc is shown in figure A.1. Thus Γ is approximable by a curve denoted by Γ' which crosses once each interval [???] of $\Gamma \cap \gamma$. This is due to the fact that all of the twists are positive. By using the criterion of proposition 3.10, we verify that $\mathrm{card}(\gamma \cap \Gamma') = i([\gamma], [\Gamma])$.

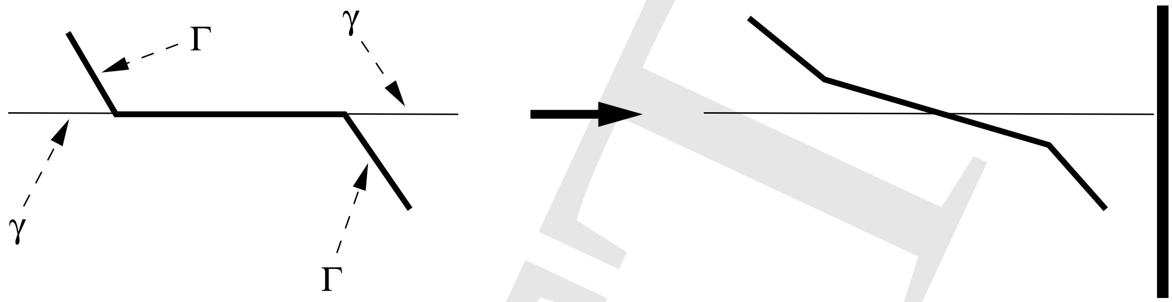


Figure A.1:

We observe that $\gamma \cap \Gamma'$ is the *image* of a continuous map, defined on $\sum_j n_j \cdot i([\gamma], [\alpha_j])$ copies of S^1, $n_j \cdot i([\gamma], [\alpha_j])$ copies of S^1 going in the free homotopy class of $[\alpha_j]$. Thus, we have the inequality

$$\mathrm{card}(\beta \cap (\gamma \cup \Gamma')) \geq \sum_j n_j \cdot i([\gamma], [\alpha_j]) \cdot i([\alpha_j], [\beta]).$$

If β does not pass through the points of intersection of γ with Γ', we have

$$\mathrm{card}(\beta \cap (\gamma \cup \Gamma')) = \mathrm{card}(\beta \cap \gamma) + \mathrm{card}(\beta \cap \Gamma').$$

If we take for β a geodesic of a metric of curvature -1 for which γ and Γ' are geodesic (such a metric exists by proposition 3.10), we have

$$\mathrm{card}(\beta \cap (\gamma \cup \Gamma')) = i([\Gamma], [\beta]) + i([\gamma], [\beta]).$$

which gives one of the desired inequalities.

It remains to prove that

$$i([\Gamma], [\beta]) \leq \sum_j n_j \cdot i([\gamma], [\alpha_j]) \cdot i([\alpha_j], [\beta]) \cdot i([\gamma], [\beta]).$$

Here we use the representative Γ rather than Γ'. We chose β in minimal position with respect to the α_j and not passing through the points of intersection of γ with α_j. Each times that β cuts α_j, β crosses the corresponding tubular neighbourhood. It thus gives $n_j \cdot i([\gamma],[\alpha_j])$ points of intersection with Γ. We therefore get

$$\mathrm{card}(\Gamma \cap \beta) = \mathrm{card}(\beta \cap \gamma) + \sum_j n_j \cdot i([\gamma],[\alpha_j]) \cdot i([\alpha_j],[\beta]).$$

If, additionally, β has minimal intersection with γ, we have $\mathrm{card}(\beta \cap \gamma) = i([\gamma],[\beta])$; the left side is always greater that or equal to $i([\Gamma],[\beta])$. ∎

Second part:

Let M be a closed surface of genus $g > 1$. Let $\mathcal{K} = \{K_1, \ldots, K_{3g-3}\}$ be a system of simple, mutually disjoint curves on M with the following properties:

1. K_j is [???];
2. If one cuts M along these curves, one obtains $(2g-2)$ pairs-of-pants (disks with two holes).

We can easily construct a simple curve α which cuts every K_j in an essential way: $i([\alpha],[K_j]) \neq 0$. Let ϕ be a diffeomorphism of M, which is equal to the identity outside of a tubular neighbourhood of α, and coincides with a single Dehn twist in the collar. We set

$$K_j' = \phi(K_j).$$

Clearly, the system $\mathcal{K}' = \{K_1', \ldots, K_{3g-3}'\}$ possesses the properties (1) and (2).

Proposition A.2 *For all j, k, we have*

$$i([K_j],[K_k']) \neq 0$$

Proof. From the inequality of proposition A.1, it follows that

$$\left| i([K_k'],[K_j]) - i([K_k],[\alpha])i([\alpha],[K_j]) \right| \leq i([K_k],[K_j]) = 0$$

∎

Remark: We may take α with $i([\alpha],[K_j]) = 2$ for all j. We then obtain $i([K_k'],[K_j]) = 4$ for all j, k.

Appendix B
Spines of manifolds of dimension 2

by V. Poénaru

Let N be a compact, connected manifold of dimension 2, with a non-empty boundary. If N is triangulated and if $L_1 \subset L_2 \subset N$ are two sub-complexes, we say that we pass from L_1 to L_2 by a dilatation of dimension n fi there exists an n-simplex τ of N and a face τ' of τ such that

$$L_2 - L_1 - \operatorname{int}\tau - \operatorname{int}\tau'$$

(here int designates the open cell.) The inverse passage is called **collapsing**. collapsing

Appendix C

Explicit formulas for Measured Foliations

by A. Fathi

Appendix D
Estimates of hyperbolic distances

by A. Fathi

Bibliography

[Abi72] W. Abikoff. *Topics in the real analytic theory of Teichmüller space*, volume 820 of *Lecture Notes in Mathematics*. Springer Verlag, New York, 1972. Cited on page(s) i

[AKM65] R.L. Adler, A.G. Konheim, and M.H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–319, 1965. Cited on page(s) 105, 108

[Ano67] D. V. Anosov. Geodesic flows on compact riemannian manifolds of negative curvature. *Proceedings of the Steklov Mathematics Institute*, 90, 1967.

[Ber78] Lipman Bers. An extemal problem for quasi-conformal mappings and a theorem by Thurston. *Acta Mathematica*, 141:73–98, 1978.

[Bir75] Joan Birman. *Braids, links, and mapping class groups*, volume 82 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1975. Cited on page(s) 18

[BO69] L. Bishop and B. O'Neill. Manifolds of negative curvature. *Trans. A.M.S.*, 145:1–48, 1969. Cited on page(s) 84

[Bou85] J. P. Bourguinon. Sur la structure du π, Séminaire de géometrie riemannienne 1970–71, variétés à courbure négative. In *Séminaire de géometrie riemannienne 1970–71, variétés à courbure négative*, 1985. Publications de l'Université Paris VII. Cited on page(s) 84

[Bow70] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, 153:401–413, 1970. Cited on page(s) 107, 108, 113, 127

[Bow78] R. Bowen. *The Structure of attractors in Dynamical Systems*, volume 668 of *Lecture Notes in Math.*, chapter Entropy and the fundamental group. Springer Verlag, 1978. Cited on page(s) 113

[BS85] F. Bonahon and L. Siebenmann. The classification of Seifert fibred 3-orbifolds. In Roger Fenn, editor, *Lecture Notes in Mathematics*, volume 95, pages 19–85. London Mathematical Society, 1985.

[CE75] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry*. North Holland, 1975. Cited on page(s) 34, 38

[Cer68] J. Cerf. *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), volume 53 of *Lecture Notes in Mathematics*. Springer Verlag, New York, 1968. Cited on page(s) 17

[CM80] H.S.M. Coxeter and W.O.J. Moser. *Generators and Relations for Discrete Groups*. Springer Verlag, Heidelberg, fourth edition, 1980.

[Cox69] H. S. M. Coxeter. *Introduction to Geometry*. John Wiley & Sons, Inc., New York, second edition, 1969.

[Din71] E.F. Dinaburg. On the relations of various entropy characteristics of dynamical systems. *Izvestija*, 35:324, 1971. English translation: Math. of the USSR, p. 337. Cited on page(s) 107

[Dun81] W. Dunbar. *Fibred orbifolds and crystallographic groups*. PhD thesis, Princeton, 1981.

[EE69] C. J. Earle and J. Eells. A fiber bundle description of Teichmüller theory. *Journal of Differential Geometry*, 3:19–43, 1969. Cited on page(s) 83

[Eps66a] D. Epstein. Curves on 2-manifolds and isotopies. *Acta Mathematica*, 115:83–107, 1966. Cited on page(s) 2, 6

[Eps66b] D. B. A. Epstein. Curves on 2-manifolds and isotopies. *Acta Mathematica*, 115:83–107, 1966. Cited on page(s) 41, 44, 45

[Eps72] D. B. A. Epstein. Periodic flows on 3-manifolds. *Annals of Mathematics*, 95:66–82, 1972.

[Fin76] R. Fintushel. Local S^1-actions on 3-manifolds. *Pacific Journal of Mathematics*, 66:111–118, 1976.

[FK98] R. Fricke and F. Klein. *Vorlesungen über die Theorie der automorphen Funktionen*. B. G. Teubner, Leibzig, 1898. Cited on page(s) 83

[FN] W. Fenchel and J. Nielsen. Discontinuous groups of non-euclidean motions. Unpublished manuscript.

[Fri76] David Fried. *Cross-sections to flows*. PhD thesis, Berkeley, 1976. Cited on page(s) 140

[Fri82a] David Fried. Flow equivalence, hyperbolic systems, and a new zeta function for flows. *Comment. Math. Helv.*, 57:237–259, 1982. Cited on page(s) 143, 144

[Fri82b] David Fried. Geometry of cross-sections to flows. *Topology*, 21:353–371, 1982. Cited on page(s) 140

[Gau27] K. F. Gauss. *Disquisitiones generales circa superfieces curvas*. Societe mathematique de France, Paris, 1827. Cited on page(s) 1

[GH78] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. Wiley Interscience, New York, 1978.

[Goo71] T.N.T. Goodman. Relating topological entropy with measure theoretic entropy. *Bull. London Math. Soc.*, 3:176–180, 1971. Cited on page(s) 127

[Gre67] Marvin L. Greenberg. *Lectures on Algebraic Topology*. W. A. Benjamin, Inc., Reading, Massachusetts, 1967.

[Gro] M. Gromov. Three remarks on geodesic dynamics and fundamental group. Preprint. Cited on page(s) 113

[GS87] Branko Grünbaum and G.C. Shephard. *Tilings and Patterns*. W. H. Freeman and Company, New York, 1987.

[Ham66] M.-E. Hamstrom. Homotopy groups of the space of homeomorphisms on a 2-manifold. *Ill. J. Math.*, 10:563–573, 1966. Cited on page(s) 143

[Har77] W. J. Harvey, editor. *Discrete groups and automorphic functions*. Academic Press, New York, 1977.

[Hem76] J. Hemple. *3-Manifolds*. Princeton University Press, Princeton, 1976. Cited on page(s) 144

[JN83] Mark Jankins and Walter D. Neumann. Lectures on Seifert manifolds. Technical Report 2, Brandeis University Mathematics Department, March 1983.

[Kau87] Louis H. Kauffman. *On Knots*. Princeton University Press, Princeton, New Jersey, 1987.

[Kel55] John L. Kelley. *General Topology*. Springer Verlag, New York, 1955.

[KR85] R. Kulkarni and F. Raymond. 3-dimensional Lorentz Space-forms and Seifert fiber spaces. Technical Report 521285, Mathematical Sciences Research Institute, 1985.

[Mal66] B. Malgrange. *Ideals of differentiable functions*. Oxford University Press, Oxford, 1966. Cited on page(s) 39

[Man75] A. Manning. Topological entropy and the first homology group. In *Dynamical Systems, Warwick 1974*, New-York, 1975. Springer Verlag. Lecture notes in Mathematics, Vol. 468. Cited on page(s) 113

[Mas67] William S. Massey. *Algebraic Topology: An Introduction*. Harcourt, Brace & World, Inc., New York, 1967.

[Mil66] John Milnor. *Topology from the differentiable viewpoint*. The University Press of Virgina, Charlottesville, 1966. Cited on page(s) 133, 136

[Mil68] J. Milnor. A note on curvature and the fundamental group. *Journal of Diff. Geometry*, 2:1–70, 1968. Cited on page(s) 110

[Mon87] José M. Montesinos. *Classical Tessellations and Three-Manifolds*. Springer Verlag, Berlin-Heidelberg, 1987.

[Nie44] J. Nielsen. Abbildungsklassen endlicher Ordnung. *Acta Mathematica*, 75:23–115, 1944.

[NS87] V.V. Nikulin and I.R. Shafarevich. *Geometries and Groups*. Springer Verlag, Berlin-Heidelberg, 1987.

[OR69] Peter Orlik and F. Raymond. On 3-manifolds with local $SO(2)$ action. *Quarterly Journal of Mathematics Oxford*, 20:143–160, 1969.

[Orl72] Peter Orlik. *Seifert Manifolds*, volume 291 of *Lecture Notes in Mathematics*. Springer Verlag, New York, 1972. Cited on page(s) 139

[Orn74] D. Ornstein. *Ergodic theory, randomness and dynamical systems*, volume 5 of *Yale Mathematical monographs*. Yale University Press, New Haven, CT, 1974. Cited on page(s) 126

[OVZ67] P. Orlik, E. Vogt, and H. Zieschang. Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten. *Topology*, 6:49–64, 1967.

[Pal60] R. Palais. Local triviality of the restriction map for embeddings. *Comm. Math. Helv.*, 34:305–312, 1960. Cited on page(s) 85

[Par64] W. Parry. Intrinsic markov chains. *Trans. Amer. Math. Soc.*, 112:55–66, 1964. Cited on page(s) 126

[Poen78] V. Poènaru. Travaux de Thurston sur les difféomorphismes des surfaces et l'espace de Teichmüller. In Roger Fenn, editor, *Séminaire Bourbaki*, volume 529. Séminaire Bourbaki, 1978. Cited on page(s) 1

[Ray68] F. Raymond. Classification of the actions of the circle on 3-manifolds. *Trans. Amer. Math. Soc.*, 131:51–78, 1968.

[Rol90] Dale Rolfsen. *Knots and Links*. Publish or Perish, Inc., Houston, Texas, 1990. Corrected Edition.

[Rou73] R. Roussarie. Plongements dans les variétés feuilletées. *Publ. Math. I.H.E.S.*, 43:143–168, 1973. Cited on page(s) 136, 139

[RS75] D. Ruelle and D. Sullivan. Currents, flows and diffeomorphisms. *Topology*, 14:319–327, 1975. Cited on page(s) 142

[RS82] C.P. Rourke and B.J. Sanderson. *Introduction to Piecewise-Linear Topology*. Springer Verlag, New York, 1982.

[Sch57] S. Schwartzman. Asymptotic cycles. *Ann. Math*, 66:270–284, 1957. Cited on page(s) 143

[Sch78] D. Schattschneider. The plane symmetry groups: Their recognition and notation. *American Mathematical Monthly*, 85:439–450, 1978.

[Sco83] Peter Scott. The geometries of 3-manifolds. *Bulletin of the London Math. Society*, 15:401–487, 1983.

[See64] R. Seeley. Extension of C^∞ functions defined in a half-space. *Proc. Amer. Math. Soc.*, 15:625–626, 1964. Cited on page(s) 39

[Sei33] H. Seifert. Topologie dreidimensionaler gefaserter Räume. *Acta Mathematica*, 60:147–288, 1933.

[Sha77] I. R. Shafaraevich. *Basic Algebraic Geometry*. Springer Verlag, New York, 1977.

[Sin76a] Y. G. Sinai. *Introduction to ergodic theory*, volume 18 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1976.

[Sin76b] Y. G. Sinai. *Introduction to ergodic theory (Math. notes)*. Princeton University Press, Princeton, 1976. Cited on page(s) 9, 126

[Sma59] Steve Smale. Diffeomorphisms of the 2-sphere. *Proc. Amer. Math. Soc.*, 10:621–626, 1959. Cited on page(s) 17

[Sma67] Steve Smale. Differentiable dynamical systems. *Bulletin of the American Mathematical Society*, 73:747–817, 1967.

[Spr66] G. Springer. *Introduction to Riemann surfaces*. Addison-Wesley, New York, 1966. Cited on page(s) 83

[ST80] H. Seifert and W. Threlfall. *A Textbook of Topology*. Academic Press, New York, 1980.

[Sta61] John Stallings. On fibering certain 3-manifolds. In *Topology of 3-manifolds and related topics*. Prentice Hall, New York, 1961. Cited on page(s) 132

[Sta71] John Stallings. *Group theory and three-dimensional manifolds*. Yale Univesity Press, Yale, 1971. Cited on page(s) 133

[Ste51] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, Princeton, New Jersey, 1951.

[Ste69] S. Sternberg. *Celestial mechanics, part II*. Benjamin, New York, 1969. Cited on page(s) 12

[Sti80] John C. Stillwell. *Classical Topology and Combinatorial Group Theory*. Springer Verlag, New York, 1980.

[Sul76] D. Sullivan. Cycles for the dynamical study of foliated manifolds and complex manifolds. *Inventiones Mathematicae*, 26:225–255, 1976.

[Thu] William Thurston. The geometry and topology of 3-manifolds. Princeton University Math. Department. Cited on page(s) 1

[Thu67] William Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin of the American Mathematical Society*, 73:747–817, 1967. Cited on page(s) 1

[Thu72] William P. Thurston. *Foliations of 3-manifolds that are circle bundles*. PhD thesis, Berkeley, 1972. Cited on page(s) 136, 139

[Thu86] William Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 59(339):i–vi and 99–130, 1986. Cited on page(s) 133, 134, 144

[Tis70] D. Tischler. On fibering certain foliated manifolds over S^1. *Topology*, 9:153–154, 1970. Cited on page(s) 131

[Wal67] F. Waldhausen. Eine Klasse von 3-dimensionalen Mannigfaltigkeiten II. *Invent. Math.*, 4:501–504, 1967.

[Wal68] F. Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. Math.*, 87:56–88, 1968. Cited on page(s) 142

[Wey55] Hermann Weyl. *Die Idee der Riemannscher Fläche*. Teubner, Leibzig, fourth edition, 1955. Cited on page(s) 6

Index