(1) Subsequences
   (a) *Nota Bene:* If $\lim_{n\to} \text{dist}(x_{n+1} - x_n) = 0$, then x_n does not necessarily converge.
   (b) E.g. consider partial sums of harmonic series: $x_n = \sum_{k=1}^{n} 1/k$.
   (c) If in addition, x_n is bounded, then x_n does not necessarily converge.
   (d) *Exercise:* Show that one can choose sequence $\epsilon_k \in \{-1, 1\}$ so that
   $$\limsup_{n\to\infty} \sum_{k=1}^{n} \epsilon_k \cdot \frac{1}{k} \;=\; 1$$
   but
   $$\liminf_{n\to\infty} \sum_{k=1}^{n} \epsilon_k \cdot \frac{1}{k} \;=\; -1.$$
   (e) But a bounded sequence of reals always has a convergent **sub**sequence.
      (i) Recall the tail $A_k = \{x_n \mid n \geq k\}$.
      (ii) For each k can find x_{n_k} so that $|x_{n_k} - \sup(A_k)| < \frac{1}{k}$.
      (iii) Show that x_{n_k} converges to $\limsup_{n\to\infty} x_n$.

(2) Sequential compactness
   (a) A is called *compact* iff every sequence has a convergent subsequence.
   (b) $\mathbb{R}$ is not compact. Let $x_n = n$. No convergent subsequence.
   (c) In $\mathbb{R}^n$: compact $\Leftrightarrow$ closed and bounded. (Heine-Borel)
   (d) $\Rightarrow$ If $\{x_n\} \subset \mathbb{R}^n$ is bounded, then it has a subsequences that converges.
   (e) For general metric spaces, closed and bounded is not enough.
      (i) *Exercise:* Show that $\mathbb{Q} \cap [0, 1]$ is closed and bounded in $\mathbb{Q}$ (with Archimedean norm), and show $\mathbb{Q} \cap [0, 1]$ is not compact.
      (ii) Define $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by
      $$d(m, n) \;=\; \begin{cases} 1 & \text{if } m \neq n \\ 0 & \text{if } m = n \end{cases}$$
         (A) *Exercise:* Show that d is a distance function.
         (B) Note that $(\mathbb{N}, d)$ is closed and bounded.
         (C) But $(\mathbb{N}, d)$ is not a compact metric space.
      (iii) Little ℓ^2:
         (A) Define ℓ^2 to be the set of sequences x_k such that the partial sums $\sum_{k=1}^{n} |x_k|^2$ converge.
         (B) Distance on ℓ^2: $d(x_k, y_k) = \sum k = 1^{\infty} |x_k - y_k|^2$. (Why is this a well-defined distance function?)

(C) The unit ball centered at zero, $B = \{x_k \mid |x_k| = 1\}$, is closed and bounded.

(D) For each n, define sequence $e_n = \{0, 0, 0, \ldots, 0, 0, 1, 0, 0 \ldots\}$ where the '1' appears in the n^{th} position.

(E) We have $\{e_n\} \subset B$, but e_n does not have a subsquence.

(F) Hence B is not compact.

(3) Lipschitz functions.

(a) A function $f : X \to Y$ is called K-Lipschitz iff

$$\operatorname{dist}(f(x), f(y)) \;\leq\; K \cdot \operatorname{dist}(x, y).$$

(b) *Exercise:* Any K-Lipschitz function is continuous.

(c) *Exercise:* If $f : [a, b] \to \mathbb{R}$ is differentiable with $|f'(x)| \leq K$, then f is K-Lipschitz.

(d) *Theorem:* Suppose that X and Y are compact. Fix K. The set of all K-Lipschitz functions is a compact subset of $C(X, Y)$.

(i) Let $\{f_n\}$ be a sequence of K-Lipschitz functions.

(ii) How to produce a subsequence that converges:

(A) Since X is compact, X contains a countable dense subset C. (See exercises under 'total boundedness' below).

(B) $\Rightarrow \exists$ sequence x_m such that for each $\forall \epsilon > 0$ and $\forall x \in X$, $\exists m$ so that $\operatorname{dist}(x, x_m) < \epsilon$.

(C) Since Y compact, $\{f_n(x_1)\}$ has a convergent subsequence. Let f_n^1 denote the associated subsequence of functions.

(D) Since Y compact, $\{f_n^1(x_1)\}$ has a convergent subsequence. Call this subsequence Let f_n^2 denote the associate subsequence of functions.

(E) Since Y compact, $\{f_n^2(x_1)\}$ has a convergent subsequence. Let f_n^3 denote the associated subsequence of functions.

(F) repeat ad infinitum

(G) Diagonalize: For each i, the sequence $f_j^j(x_i)$ converges (why?) and hence is Cauchy.

(iii) Claim: for each $x \in X$, the sequence $\{f_j^j(x)\}$ is Cauchy.

(A) Let $\epsilon > 0$.

(B) Choose m so that $|x_m - x| < \epsilon/(3K)$

(C) Choose M so that $i, j > M \Rightarrow |f_i^i(x_m) - f_j^j(x_m)| < \epsilon/3$.

(D) Estimate: Using triangle inequality and K-Lipschitz

$$\begin{aligned} |f_i^i(x) - f_j^j(x)| &\leq |f_i^i(x) - f_i^i(x_m)| + |f_i^i(x_m) - f_j^j(x_m)| + |f_j^j(x_m) - f_j^j(x)| \\ &\leq K \cdot |x - x_m| + |f_i^i(x_m) - f_j^j(x_m)| + K \cdot |x_m - x| \\ &\leq \epsilon. \end{aligned}$$

(iv) Let $f(x)$ be the limit of $\{f_j^j(x)\}$

(v) Since $|f_j^j(x) - f_j^j(y)| \leq K|x-y|$ we have $|f(x) - f(y)| \leq K|x-y|$. So f is Lipschitz and hence continuous.

(4) Compactness via covers

(a) (X, dist) metric space

(b) The (open) *ball* of radius r centered at x is the set

$$\{y \in X \mid \text{dist}(x, y) < r\}.$$

Notation: $B(x, r)$.

(c) Let $A \subset B$. A *covering* of A is a collection of balls in X whose union contains A.

(d) A set A is *compact* iff given any covering of A, we can remove all but finitely many of the balls, and still cover A. ('finite subcover')

(5) Relation between compactness and completeness

(a) Compactness $\Rightarrow$ completeness

(i) Let x_n be a Cauchy sequence.

(ii) Compactness $\Rightarrow$ a convergent subsequence has limit.

(iii) *Exercise:* Show that x_n converges to this limit.

(iv) Converse is not true: $\mathbb{R}$ is complete but not compact.

(b) Total boundedness

(i) Special cover: The ϵ-*cover* of A consists of all balls $B(x, \epsilon)$, with $x \in A$.

(ii) A is totally bounded iff $\forall \epsilon > 0$, the ϵ-cover of A has a finite subcover.

(iii) *Exercise:* Any bounded subset of $\mathbb{R}^n$ is totally bounded.

(iv) *Exercise:* If A is compact, then A is totally bounded.

(v) *Exercise:* If A is totally bounded, then there exists a countable subset $C \subset A$ that is dense in A. (We say that C is *dense* in A, if for every $\epsilon > 0$ the union of all balls $B(c, \epsilon)$ where $c \in C$ contains A.)

(c) *Theorem:* X is compact if and only if X is both complete and totally bounded.

(6) Arzela-Ascoli theorem (one variation)

(a) $F \subset C(X,Y)$ is called *equicontinuous* iff $\forall \epsilon > 0$ and $\forall x \in X$, there exists $\delta_{x,\epsilon}$ such that $\forall f \in F$, we have

$$\text{dist}(x,x') < \delta_{x,\epsilon} \Rightarrow \text{dist}(f(x), f(x')).$$

(b) *Exercise:* Fix K, and let X and Y be compact metric spaces. Show that the set of K-Lipschitz functions in $C(X,Y)$ is equicontinuous.

(c) *Exercise:* Define $f_n : [0,1] \to [0,1]$ by $f_n(x) = x^n$. Show that $\{f_n\}$ is not equicontinuous.

(d) *Theorem:* Let X and Y be compact. If $F \subset C(X,Y)$ is equicontinuous, then F is totally bounded.

(e) *Corollary:* (Arzela-Ascoli) Let X be compact, and suppose that $\{f_n\}$ is a sequence that belongs an equicontinuous subset of $C(X,\mathbb{R})$. If there exists M such that $|f_n(x)| \leq M$ for all n and x, then f_n has a convergent subsequence.

(7) Application: Euler's approximation method apres Peano

(a) Want to solve $y'(t) = F(t, y(t))$ with $y(0) = 0$ and F continuous (but not necessarily Lipschitz in second variable).

(b) Euler approximation

(i) An 'Euler approximant' is a continuous piecewise linear function that results from the following process

(ii) Pick $t_1 > 0$. (Here we set $t_0 = 0$.)

(iii) Define y on $[t_0, t_1]$ to be the linear function with $y(t_0) = 0$ and slope equal to $F(t_0, y(t_0))$.

(iv) Pick $t_2 > t_1 = 0$

(v) We extend y to the unique continuous function on $[0, t_2]$ that is linear on $[t_1, t_2]$ slope equal to $F(t_1, y(t_1))$.

(vi) Repeat (i.e. pick $t_3 > t_2$ etc.)

(vii) y is the *Euler approximant* associated to the sequence $0 = t_0 < t_1 < t_2 < t_3 \cdots$.

(viii) Idea: The approximation becomes better if the $\sup\{|t_{i+1} - t_i|\}$ becomes smaller.

(c) Equicontinuity of the set of Euler approximations.

(i) $M = \sup\{|F(t,y)| \mid (t,y) \in [0,1] \times [-1,1]\}$

(ii) The restriction of each Euler approximations to $[0, 1/M]$ is M-Lipschitz.

(A) If $t_i < s \leq t < t_{i+1}$, then

$$|y(s) - y(t)| \ \leq \ M|s-t|.$$

(B) If $s \le t_i < \cdots < t_j \le t$, then

$$
\begin{aligned}
|y(s) - y(t)| &\le |y(t_i) - y(s)| + \sum_{k=i}^{j} |y(t_{k+1}) - y(t_k)| + |y(t) - y(t_j)| \\
&\le M\left(|t_i - s| + \sum_{k=i}^{j} |t_{k+1} - t_k| + |t - t_j|\right) \\
&\le M|s - t|.
\end{aligned}
$$

(d) Now apply Arzela-Ascoli to get convergent subsequence.

(e) Separate argument shows that limit is a solution to ODE.

(f) Solution is not necessarily unique; different subsequences.

2. Additional exercises

(1) Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of differentiable functions such that for each $x \in [0, 1]$,

  (a) $f_n(x)$ converges to 0 (ptwise convergence)

  (b) and $|f_n'(x)| \le 1$.

  Then f converges to the 0 function in $C([0, 1], \mathbb{R})$. (uniform convergence)

(2) Let $G(x, y)$ be a continuous function on $\mathbb{R}^2$ and suppose for each positive integer k, that g_k is a continuous function defined on $[0, 1]$ with the property that $|g_k(y)| < 1$ for all $y \in [0, 1]$. Now define

$$f_k(x) := \int_0^1 g_k(y) \cdot G(x, y) dy.$$

  Prove that the sequence $\{f_k\}$ is equicontinuous on $[0, 1]$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $x \in \mathbb{R}$ and define the sequence $\{x_n\}_{n=0}^{\infty}$ inductively by setting $x_0 = x$ and $x_{n+1} = f(x_n)$. Suppose that $\{x_n\}$ is bounded. Prove that there exists $y \in \mathbb{R}$ such that $f(y) = y$. *(Suggestion: Consider function $f(x) - x$ and use intermediate value theorem and monotone convergence theorem.)*