1. CONTRACTIONS AND COMPLETENESS

(1) Newton's method:
   (a) Want to solve $g(x) = 0$.
   (b) Make initial guess x_0.
   (c) Tangent line approximation: $y - g(x_0) = g'(x_0)(x - x_0)$.
   (d) Set $y = 0$ and solve for x to obtain 'better guess':
$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}.$$
   (e) In general, set $f(x) = x - \frac{g(x)}{g'(x)}$, and $x_{n+1} = f(x_n)$.
   (f) If $g(x) = x^2 - 2$, then we get the Babylonian sequence.
   (g) Do we always get convergence?

(2) Differential equations
   (a) Major focus of last 100 years
   (b) Iteration methods (Picard, Nash, Moser)

(3) Picard iteration
   (a) Consider ODE $y'(t) = F(t, y(t))$.
   (b) Equivalent to: $y(t) = y(0) + \int_0^t F(s, y(s))\, ds$. (Fun thm of calculus)
   (c) Solution y is a fixed point of a mapping ϕ.
      (i) Namley, define ϕ by
$$(\phi(y))(t) = y(0) + \int_0^t F(s, y(s))\, ds.$$
      (ii) Note that ϕ well-defined if y and F are continuous.
      (iii) Then y is a solution to (b) if and only if $\phi(y) = y$.
   (d) Iterate to find fixed point:
      (i) Make initial guess y_0.
      (ii) Set $y_{n+1} = \phi(y_n)$

(4) Does y_n converge to fixed point of ϕ?
   (a) Idea: limit is fixed point.
   (b) **Assume** that $\exists\, K$ so that $|F(s,y) - F(s,y')| \leq K \cdot |y - y'|$. (Lipschitz).
   (c) The estimate:
$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq \int_0^t |F(s, y_n(s)) - F(s, y_{n-1}(s))|\, ds \\ &\leq K \cdot |t - 0| \sup_{s \in [0,t]} |y_n(s) - y_{n-1}(s)|. \end{aligned}$$

(d) Let $T = 1/(2K)$ and define $|f| = \sup\{f(t) \mid t \in [0, T]\}$.

(e) Then the estimate can then be interpreted as
$$|y_{n+1} - y_n| \leq \frac{1}{2} \cdot |y_n - y_{n-1}|.$$

(f) Analogy with Babylonan sequence leads one to believe that this sequence is 'Cauchy'.

(g) But

(i) how does one define 'Cauchy' in this context?

(ii) do such 'Cauchy' sequences converge?

(iii) and if so, is the limit a continuous function?

(5) Cauchy sequences in metric spaces

(a) A metric or distance function on X is a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ that satisfies

(i) $d(x, y) = d(y, x)$

(ii) $d(x, z) \leq d(x, y) + d(y, z)$

(iii) $d(x, y) = 0$ iff $x = y$

(b) Cauchy sequences and completeness

(i) The sequence x_n is said to be *Cauchy* with respect to the metric d iff $\forall m$, $\exists M$ so that $i, j > M \Rightarrow d(x_i, x_j) < 1/m$.

(ii) The metric space (X, d) is *complete* iff every Cauchy sequence converges.

(c) Contraction mapping principle

(i) *Theorem:* If X is a complete metric space, and $\phi : X \to X$ satisfies
$$d(\phi(x), \phi(x')) \ \leq \ \frac{1}{2} \cdot d(x, x')$$
for all $x \in X$, then ϕ has fixed point.

(ii) Can replace $1/2$ with any constant strictly less than 1.

(d) Example: continuous functions with 'sup norm'

(i) Let $C([a, b], \mathbb{R})$ denote the set of all continuous functions $f : [a, b] \to \mathbb{R}$.

(ii) For each $f, g \in C([a, b], \mathbb{R})$, define the distance between f and g to be
$$d(f, g) = \sup\{|f(t) - g(t)| \mid t \in [a, b]\}.$$

(iii) This is called the *sup norm* or (L^∞ *norm*) distance.

(iv) Terminology: If $\lim_{n\to\infty} \mathrm{dist}(f_n, f) = 0$, we say that f_n *converges uniformly.*

(6) *Theorem:* $C([a,b],\mathbb{R})$ is complete with respect to the sup norm distance.

(a) Let f_n be a sequence in $C([a,b],\mathbb{R})$ that is Cauchy with respect to sup norm.

(b) In other words, for all $\epsilon > 0$, $\exists M$ so that if $m, n > M$ then

$$\sup\{|f_n(t) - f_m(t) \mid t \in [a,b]\} < \epsilon.$$

(c) Existence of a limit:

(i) Fix x.

(ii) Note that $\{f_n(x)\} \subset \mathbb{R}$ is Cauchy with respect to the usual distance on $\mathbb{R}$.

(iii) Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ has a limit.

(iv) Define $f(x)$ to be this limit.

(v) f is called the *pointwise limit* of the sequence f_n.

(d) Convergence with respect to sup norm:

(i) Let $\epsilon > 0$.

(ii) Choose M so that $i, j > M \Rightarrow \sup_x |f_i(t) - f_j(t)| < \epsilon$.

(iii) Fix $j > M$.

(iv) For all $i > M$ and all $t \in [a,b]$ we have $|f_i(t) - f_j(t)| < \epsilon$.

(v) Note that for each given y, the function $g(x) = |x - y|$ is a continuous function

(vi) Hence, for each $t \in [a,b]$

$$|f(t) - f_j(t)| = \lim_{i\to\infty} |f_i(t) - f_j(t)| \ \leq \ \epsilon/3$$

(vii) In other words, for each $j > M$, we have $d(f, f_j) < \epsilon$.

(viii) Thus, f_j converges to f with respect to sup norm.

(e) The limit is continuous:

(i) Fix $x \in [a,b]$ and $\epsilon > 0$.

(ii) Since $d(f_j, f) \to 0$, we can choose j so that if $t \in [a,b]$, then

$$|f_j(t) - f(t)| \ < \ \epsilon/3.$$

(iii) Fix this j.

(iv) Since f_j continuous, $\exists\ \delta > 0$ so that if $|x - y| < \delta$, then

$$|f_j(x) - f_j(y)| \ < \ \epsilon/3.$$

(v) Triangle inequality $\Rightarrow \forall y$ such that $|x-y|<\delta$ we have

$$\begin{aligned} |f(x)-f(y)| &\leq |f(x)-f_j(x) \ + \ f_j(x) \ - \ f_j(y) \ + \ f_j(y) \ - \ f(y)| \\ &\leq |f(x)-f_j(x)|+|f_j(x)-f_j(y)|+|f_j(y)-f(y)| \\ &\leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}. \end{aligned}$$

(vi) Thus, f is continuous.

(f) More general situation:

(i) Let X be a compact topological space and let (Y, d_Y) be a metric space.

(ii) Let $C(X, Y)$ be the continuous functions from X to Y.

(iii) Define $d : C(X, Y) \times C(X, Y) \to \mathbb{R}$ by

$$d(f, g) \ = \ \sup \{d_Y(f(x), g(x)) \ | \ x \in X\}.$$

(iv) Since X compact, d is well-defined.

(v) If, for example, $X = \mathbb{R}$, then $f(x) = x$ does not have finite distance from $g(x) = 0$.

(vi) *Exercise:* Show that d is a distance function.

(vii) *Theorem:* If Y is complete, then $C(X, Y)$ is complete.

(viii) The proof is the same!

(7) Closedness and Completeness.

(a) Let (X, d) be a metric space, and let $A \subset X$. A point $x \in X$ is called a *limit point* of $A \iff$ there exists a sequence $\{a_n\} \subset A$ so that $\lim a_n = x$.

(b) A subset $A \subset X$ is called *closed* $\iff$ all of the limit points of A are contained in A.

(c) Exercise: Show that a closed subset A of a complete metric space X is complete, i.e. if $\{a_n\} \subset A$ is Cauchy then a_n has a limit.

(d) Application: Babylonian sequence converges

(i) $[1, \infty)$ is complete

(ii) $g(x) = x/2 + 1/x$ maps $[1, \infty)$ into $[1, \infty)$

(iii) Thus, since g is a contraction, g has a fixed pt in $[1, \infty)$.

1.1. **additional exercises.**

(1) The astronomer Halley proposed the following iterative scheme for solving $f(x) = 0$.

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x)^2 - f(x_n)f''(x_n)}.$$

Does Halley's method converge?

(2) Let I be a closed interval in R, and let f be a differentiable real valued function on I, with $f(I) \subset I$. Suppose that $|f'(t)| < 3/4$ for all $t \in I$. Let x_0 be any point in I and define a sequence x_n by $x_{n+1} = f(x_n)$ for every $n > 0$. Show that there exists $x \in I$ with $f(x) = x$ and $\lim x_n = x$.

(3) Define $\Phi : C([a,b]) \to C([a,b])$ by

$$[\Phi(f)](t) \;=\; 1 + \int_0^t s^2 e^{-f(s)}\, ds.$$

Define $f_0 \equiv 1$ and $f_{n+1} = \Phi(f_n)$.

(a) Prove that $1 \leq f_n(t) \leq 1 + 1/3$

(b) Prove that

$$|f_{n+1}(x) - f_n(x)| < \frac{1}{3} \sup\{|f_n(t) - f_{n-1}(t)| \mid 0 \leq t \leq 1\}$$

(c) Show that f_n converges to f in $C([a,b])$.