1. Mappings that are almost linear

(1) Differentiability in 1-dimension.

(a) $f : \mathbb{R} \to \mathbb{R}$ is said to be *differentiable at* $x \in \mathbb{R}$ iff

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

exists.

(b) The limit (if it exists) is called the *derivative of* f *at* x and is denoted by $f'(x)$.

(c) Exercise: Show that f differentiable at $x \Rightarrow f$ continuous at x. (Similar to proof that K-Lipschitz implies continuous.)

(d) *Nota Bene:* differentiability does not imply K-Lipschitz.
E.g. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is not K-Lipschitz for any K.

(e) Example: $\exists$ a function that is continuous everywhere but differentiable nowhere.

(i) Idea: Construct a sequence of piecewise linear functions f_n using inductive procedure, and then take limit. (Bolzano)

(ii) Define f_1 by $f_1(x) = x$.

(iii) Suppose that f_n has been constructed.

(iv) Let $[a, b]$ be a (maximal) interval of increase (or decrease) of f_n.

(v) Divide $[a, b]$ into 3 subintervals $[a, a']$, $[a', b']$ and $[b', b]$ of equal length.

(vi) f_{n+1} is uniquely defined by the following conditions:

(A) The restriction of f_{n+1} to each of the three subintervals is linear.

(B) $f_{n+1}(a) = f_n(a)$

(C) $f_{n+1}(b) = f_n(b)$

(D) $f_{n+1}(a') = f_n(a) + \frac{3}{4}(f_n(b) - f_n(a))$

(E) $f_{n+1}(b') = f_n(b) - \frac{3}{4}(f_n(b) - f_n(a))$

(vii) Continuity everywhere:

(A) *Exercise:* Show that f_n is Cauchy in $C^0([0, 1], \mathbb{R})$.

(B) Thus since $C^0([0, 1], \mathbb{R})$, f_n has continuous limit f.

(viii) Differentiability nowhere:

(A) Need exercise: If f is differentiable at x, $a_n \nearrow x$, and $b_n \searrow x$, then

$$\lim_{n \to \infty} \frac{f(a_n) - f(b_n)}{a_n - b_n} = f'(x).$$

(B) Note that if $n \geq m$ and all k we have $f(\frac{k}{3^m}) = f_n(\frac{k}{3^m})$.

(C) Use this to show that

$$\left| \frac{f(\frac{k+1}{3^m}) - f(\frac{k}{3^m})}{\frac{k+1}{3^m} - \frac{k}{3^m}} \right| \geq (3/2)^m.$$

(D) Use exercise to conclude that f is not differentiable at every x.

(2) Mean value theorem (MVT)

(a) *Theorem:* Let $f : [a, b]$ be differentiable on (a, b) and continuous on $[a, b]$, then $\exists c \in (a, b)$ so that

$$f(b) - f(a) = f'(c) \cdot (b - a).$$

(b) Often the MVT can be used to get slightly better than expected.

(c) Application: IVT for derivative

(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ differentiable and we know that $f'(a) > m > f'(b)$.

(ii) What else do we need to know about f in order to guarantee that $\exists c \in (a, b)$ so that $f'(c) = m$?

(iii) Answer: Nothing more!

(A) Definition of $f'(a)$ and $f'(b)$ implies $\exists\ h > 0$ so that

$$\frac{f(a+h) - f(a)}{h} \ < \ y \ < \ \frac{f(b) - f(b-h)}{h}.$$

(B) Fix h.

(C) Note that

$$g(t) = \frac{f(t+h) - f(t)}{h}$$

is defined and continuous on $[a, b - h]$.

(D) By IVT, $\exists\ c \in (a, b - h)$ so that $g(c) = m$.

(E) In other words:

$$\frac{f(c+h) - f(c)}{h} \ = \ m.$$

(F) On the other hand, MVT gives $d \in (c, c + h)$ so that

$$f(c+h) - f(c) \ = \ f'(d) \cdot h.$$

(G) Thus, $f'(d) = m$.

(3) Definition of derivative in higher dimensions.

(a) If $n = 1$ and m is arbitrary, then the same formula works since h is scalar.

(b) But if n larger, then it doesn't make sense to divide vectors.

(c) The directional derivative.

(i) Fix $p \in \mathbb{R}^n$.

(ii) The *directional derivative* in the direction $v \in \mathbb{R}^n$ is defined to be

$$df_p(v) = \lim_{h \to 0} \frac{1}{h} \left(f(p + hv) - f(p) \right).$$

(iii) Alternately:

$$df_p(v) = \left. \frac{d}{dt} \right|_{t=0} f(p + tv).$$

(iv) The directional derivative should be regarded as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(d) Recall that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* iff

$$T(ax + by) = T(ax) + T(by)$$

for all $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.

(e) f is differentiable at p iff

(i) For each v, the directional derivative $df_p(v)$ exists,

(ii) $v \to df_p(v)$ is linear, and

(iii) $\forall \epsilon > 0$, $\exists \delta > 0$ so that if $|w| < \delta$ then

$$\left| \frac{f(p + w) - f(p)}{|w|} \; - \; df_p\left(\frac{w}{|w|} \right) \right| \; < \; \epsilon.$$

(f) Dimension 1 revisited: $f'(p) = df_p(1)$ where 1 is the usual unit vector in $\mathbb{R}$.

(g) *Exercise:* Show that if f is differentiable at p, then f is continuous at p.

(4) Constructing examples of functions that are **not** differentiable at p.

(a) Without loss of generailty $p = 0$.

(b) Using polar coordinates:

(i) $x = r\cos(\theta)$ and $y = r\sin(\theta)$

(ii) Consider functions of the form $f(r, \theta) = r^k \cdot g(\theta)$.

(iii) *Nota Bene:* If $k \geq 2$, then $\forall v$ we have $df_0(v) = 0$.

(c) Example: Conditions (i) and (ii) of the definition do not imply (iii):

   (i) Let $k = 2$ and $g(\theta) = 1/\theta$ for $0 < \theta \leq 2\pi$.
   (ii) Then $df_p(v) = 0$ for all v and hence df_p is linear.
   (iii) Consider f along curve $\theta = \frac{1}{r^2}$.
   (iv) For $r > 0$ we have $f(r, 1/r^2) \equiv 1$.
   (v) But $f(0)$, and hence f is not continuous at 0.

(d) Example: Condition (i) of definition does not imply condition (ii).
   (i) $f : \mathbb{R}^2 \to \mathbb{R}$ described in polar coordinates by
   $$f(r, \theta) \;=\; r \cdot g(\theta)$$
   where $g(\theta) = \sin(3\theta)$.
   (ii) No plane tangent to the graph of f.
   (iii) In rectangular coordinates
   $$f(x, y) = \frac{3xy^2 + y^3}{x^2 + y^2}$$
   with $f(\vec{0}) = 0$.
   (iv) Verfication that f not differentiable at $\vec{0} = (0, 0)$:
      (A) If df_0 were linear, then $df_p(x + y) = df_p(x) + df_p(y)$ for all x and y.
      (B) Let $x = (1, 0)$ and $y = (0, 1)$.
      (C) We have
      $$df_0(x) \;=\; \frac{d}{dt}\bigg|_{t=0} f((t, 0)) = \frac{d}{dt}\bigg|_{t=0} 0 = 0,$$
      (D) and
      $$df_0(y) \;=\; \frac{d}{dt}\bigg|_{t=0} f((0, t)) = \frac{d}{dt}\bigg|_{t=0} t = 1,$$
      (E) but
      $$df_0(x + y) \;=\; \frac{d}{dt}\bigg|_{t=0} f((t, t)) = \frac{d}{dt}\bigg|_{t=0} 2t = 2,$$
      (F) Since $2 \neq 1 + 0$, f is not differentiable at $\vec{0}$.

(5) MVT is false in higher dimensions, but
(6) Mean value type estimate in all dimensions:
   (a) Given T linear, define $||T|| = \sup_{v \neq 0} \frac{|T(v)|}{|v|}$
   (b) Suppose that $||df_p|| \leq M$ for all p along the segment joining a and b, then $|f(b) - f(a)| \leq M \cdot |b - a|$
   (c) Pf: Apply fun thm of calculus. (In 1-D comes from MVT).

(7) Partial derivatives and Jacobian matrix

(a) What is practical purpose of condition (b) in the definition of the derivative? Answer: Want to reduce analysis of functions to linear algebra.

(b) Let e_i is unit vector pointing in direction of positive x_i axis. The set $\{e_i\}$ is called the *standard basis.*

(c) Choice of basis allows us to express linear transformation T with only finitely many numbers:

(i) Every vector v can be written uniquely as linear combination of basis vectors $v = a_1 e_1 + \cdots + a_n e_n$.

(ii) T linear $\Rightarrow T(v) = a_1 T(e_1) + \cdots + a_n T(e_n)$.

(iii) If we regard $T(e_i)$ as the columns of a matrix A, then $T(v) = A \cdot v$.

(d) The matrix associated to df_p is called the *Jacobian matrix* of f at p.

(e) Partial derivatives

(i) Suppose $f : \mathbb{R}^n \to \mathbb{R}$. The j^{th} partial derivative at p is

$$\partial_j f(p) = \frac{\partial f}{\partial x_j}(p) = df_p(e_j).$$

(ii) The (i, j) entry of the Jacobian matrix is the j^{th} partial derivative of the i^{th} coordinate function. (Jacobian matrix is also called "matrix of first partials").

(f) *Theorem:* If partials $\partial_i f(x)$ are defined in a ball $B(p, r)$ and are continuous at p, then f is differentiable at p.

(g) Existence of partials is not enough. Indeed, above we gave an example of a function such that all directional derivatives existed but function not differentiable. A partial derivative is just a special directional derivative.

(8) Chain rule

(a) Suppose f differentiable at p and g differentiable at $g(p)$. Then $g \circ f$ differentiable at p and

$$d(g \circ f)_p = dg_{g(p)} \circ df_p.$$

(b) Thus, if A is Jacobian matrix for f at p and B is Jacobian matrix for g at $f(p)$, then the Jacobian matrix for $g \circ f$ is $B \cdot A$.

1.1. **Additional Exercises.**

(1) Define $f : [0, 1] \to \mathbb{R}$

$$f(x) \;=\; \begin{cases} x^2 \sin(\pi/x^2), & x > 0 \\ 0, & x = 0. \end{cases}$$

(a) Check that f is differentiable at each $x \in [0,1]$ including $x = 0$.

(b) Show that f is not K-Lipschitz for any K.

(c) Show that $\limsup_{x\to 0} |f'(x)| = \infty$.

(2) Let $n \in \mathbb{N}$ and let $t_0 \in [a,b]$. Suppose that the n^{th} derivative $f^{(n)}$ exists and is identically zero in the interval $[a,b]$ and that $0 = f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0)$. Show that f must be idetically zero.

(3) Let df_p be the derivative at p of the differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that $\exists C$ so that $|f(x) - f(y)| > C|x - y|$ for all x, y. Show that for each p, the linear transformation df_p is an invertible.