1. The fundamental theorem of calculus

(1) Classical version

(a) If $f : [a, b] \to \mathbb{R}$ is a differentiable function whose derivative is integrable, then

$$\int_a^b f' = f(b) - f(a).$$

(b) Note: Each side may be regarded as a sum.

(i) The left-hand side is a limit of Riemann sums.

(ii) The right hand sum is a weighted sum.

(c) We wish to generalize this formula to higher dimensions.

(2) Differential forms

(a) In what follows I will denote a k-tuple

$$I = (i_1, i_2, \ldots, i_k)$$

where $\forall j$ we have $i_j \in \{1, \ldots, n\}$, .

(b) The *basic k-form* on $\mathbb{R}^n$ associated to I is defined to be

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

(c) A differential k-form is a linear combination (over the ring $C^\infty(\mathbb{R}^d)$) of basic k-forms

$$\sum_I f_I \cdot dx_I.$$

(d) A differentiable function is called a 0-form.

(e) We require antisymmetry: $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

(f) exterior derivative:

(i) If f is a function (i.e. a 0-form)

$$df = \sum_{j=1}^{d} \frac{\partial f}{\partial x_j} dx_j.$$

(ii) If $k > 0$, then

$$d\left(\sum_I f_I \cdot dx_I\right) = \sum_I df_I \wedge dx_I.$$

(g) pull-back:

(i) Suppose $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

(ii) The *pull-back* of the basic k-form dx_I is

$$\phi^*(dx_I) \;=\; d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}$$

where ϕ_j is the j^{th} coordinate function of ϕ.

(iii) In general

$$\phi^*\left(\sum_I f_I \cdot dx_I\right) \;=\; \sum_I (f_I \circ \phi) \cdot \phi^*(dx_I).$$

(3) Integration of differential n-forms on $\mathbb{R}^n$

(a) Orientation

(i) Let $k = n$.

(ii) We say that I is *positively oriented* if and only if

$$dx_I \;=\; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

(iii) In other words, I is positively oriented iff the permutation

$$(1, 2, \ldots, n) \mapsto (i_1, i_2, \ldots, i_n)$$

is even.

(b) Integration of n-forms on $\mathbb{R}^n$

(i) If I is positively oriented, then define

$$\int_U f_I \cdot dx_I \;=\; \int_U f_I$$

where $\int_U f_I$ is Riemann integral over U.

(ii) If I is not positively oriented, then define

$$\int_U \sum_I f_I \cdot dx_I \;=\; -\int_U f_I$$

(iii) Extend to be linear:

$$\int_U \sum_I f_I \cdot dx_I \;=\; \sum_I \int_U f_I \cdot dx_I.$$

(iv) Note that definition depends on integrability properties of f_I over U. Usually assume that f_I are smooth and the boundary ∂U has content zero (e.g. piecewise smooth).

(c) Change of variables formula:

(i) Let ω be n-form.

(ii) Let U, $V \subset \mathbb{R}^n$ be open sets with piecewise smooth boundary.

(iii) Suppose that $\phi : V \to U$ satisfies

(A) ϕ is bijection

  (B) $d\phi_p$ is invertible for every $p \in V$,
  (C) ϕ is *orientation preserving*: $\forall p$, we have $\det(d\phi_p) > 0$,
(iv) Then for each differential n-form ω

$$\int_V \phi^*(\omega) \;=\; \int_U \omega.$$

(4) Integrating differential forms over parametrized submanifolds.
  (a) Let $U \subset \mathbb{R}^k$.
  (b) Let $\gamma : U \to \mathbb{R}^n$ be an injective immersion.
  (c) The image $\gamma(U)$ is a parametrized k-dimensional (immersed) submanifold of $\mathbb{R}^n$.
  (d) Let ω be a differential k-form on $\mathbb{R}^n$.
  (e) *Definition:* The integral of ω over the parameterization γ is defined to be

$$\int_\gamma \omega \;:=\; \int_U \gamma^*(\omega).$$

  (f) Orientation
    (i) Suppose that $\alpha : A \to \mathbb{R}^n$ and $\beta : B \to \mathbb{R}^n$ both parameterize $M \subset \mathbb{R}^n$.
    (ii) We say that α and β determine the same orientation on M iff $\alpha \circ \beta^{-1}$ is orientation preserving.
    (iii) Equivalence:
      (A) Write $\alpha \sim \beta$ if two injective immersions determine the same orientation.
      (B) This is an equivalence relation on parameterizations of M.
      (C) *Exercise:* Show that there are exactly two equivalence classes.
      (D) Each equivalence class is called an *orientation* of the parameterized submanifold M.
    (iv) If $\alpha \sim \beta$, then

$$\int_\alpha \omega \;=\; \int_\beta \omega.$$

    (v) Otherwise

$$\int_\alpha \omega \;=\; -\int_\beta \omega.$$

    The image $\gamma(U)$ is a parametrized k-dimensional (immersed) submanifold of $\mathbb{R}^n$.

(g) Definition of intergral over oriented submanifold.

(i) Let M be an oriented parameterized submanifold.

(ii) Let γ belong to the orientation class.

(iii) Define

$$\int_M \omega \;=\; \int_\gamma \omega.$$

(5) The modern Stokes theorem

(a) open subsets of $\mathbb{R}^n$

(i) Let U be an open subset of $\mathbb{R}^n$ with $n-1$ dimensional boundary ∂U.

(ii) Let $n : \partial U \to \mathbb{R}^n$ be the outward normal vector field.

(iii) Suppose that $\gamma : V \to \mathbb{R}^n$ be a parameterization of ∂U.

(iv) A parameterization belongs to the *outward orientation* iff for each $p \in V$, the matrix

$$[n(p), d\phi_p(e_1), d\phi_p(e_2), \cdots, d\phi(e_{n-1})].$$

has positive determinant.

(v) Stokes theorem: For each $n-1$ form ω

$$\int_{\partial U} \omega \;=\; \int_U d\omega$$

where ∂U has its outward normal orientation.

(b) Parameterized manifolds with boundary.

(i) Let $M \subset R^n$ be an oriented parameterized k-dimensional submanifold with $k-1$-dimensional parameterized boundary ∂M.

(ii) Let $\gamma : U \to \mathbb{R}^n$ be a parameterization of M in the orientation class of M.

(iii) A parameterization $\alpha : V \to \mathbb{R}^n$ of ∂U is compatible with the orientation of M if and only if $\gamma^{-1} \circ \alpha$ lies in the outward normal orientation class of ∂U.

(iv) Associated to the orientation class of M, there exists a unique compatible orientation class of ∂M.

(v) Stokes Theorem: For each $n-1$ form ω

$$\int_{\partial M} \omega \;=\; \int_M d\omega$$

where ∂M has the orientation induced by M.

(6) The module of differential k-forms.

(a) Let Ω^k denote the differential k-forms.

(b) algebraic structure

(i) addition

(ii) 0 is additive identity

(iii) multiplication by smooth functions

(iv) wedge product (exterior algebra)

(c) basis and dimension

(i) Note: If ω is a k-form on $\mathbb{R}^n$ and $k > n$, then $\omega = 0$.

(ii) Assume $k \leq n$.

(iii) $I = (i_1, i_2, \ldots, i_k)$ is said to be *ordered* if anf only if $r < s \Rightarrow i_r < i_s$.

(iv) *Exercise:* The number of ordered k-tuples in $\{1, \ldots, n\}^k$ is $\binom{n}{k}$.

(v) For each k-form ω and ordered k-tuple I there exists a unique function f_I so that

$$\omega = \sum_{I \text{ ordered}} f_I \cdot dx_I.$$

(vi) Ω^k is an $\binom{n}{k}$-dimensional module over the ring of smooth functions.

(7) Functions and differential forms

(a) Let $C^\infty = C^\infty(U)$ denote the space of smooth functions on U.

(b) 0-forms are functions: $\Omega^0 = C^\infty$

(c) ω is n-form iff $\omega = f \cdot dx_1 \wedge \cdots \wedge dx_n$.

(d) Module isomorphism $\Phi_n : \Omega^n \to C^\infty$ defined by

$$\Phi_n(f \cdot dx_1 \wedge \cdots \wedge dx_n) = f.$$

(8) Vector fields and differential forms

(a) Let $U \subset \mathbb{R}^n$.

(b) A vector field X on U is a function $X : U \to \mathbb{R}^n$.

(c) Let $\mathcal{V}$ be the set of vector fields on U.

(i) addition

(ii) multiplication by smooth functions $\mathbb{R}^n \to \mathbb{R}$.

(iii) Let e_i denote the the i^{th} standard vector field.

(iv) Standard vectors fields provide basis over smooth functions.

(d) Module isomorphsim $\Phi_1 : \Omega^1 \to \mathcal{V}$ defined by

$$\Phi_1\left(\sum_i f_i \; dx_i\right) = \sum_i f_i \cdot e_i.$$

(e) Module isomorphism Φ_{n-1}

(i) For each j, let I_j be the $n-1$-tuple $(1, 2, \cdots, j-1, j+1, \ldots, n)$.

(ii) For each $n-1$-form ω and each $j \in \{1, \ldots, n\}$ there exists unique $f_j \in C^\infty$ such that

$$\omega = \sum_j f_j \cdot dx_{I_j}.$$

(iii) Define $\Phi_{n-1} : \Omega^{n-1} \to \mathcal{V}$ by

$$\Phi_n \left(\sum_j f_j \cdot dx_{I_j} \right) = \sum_j f_j \cdot e_j.$$

(iv) flux form.

(9) div, grad, curl, and the exterior derivative

(a) The classical integration theorems—Greens, Stokes, divergence—follow from modern Stokes theorem.

(b) Need only reinterpret the following operators in terms of d.

(c) gradient

(i) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

(ii) The gradient of f is the vector field

$$\operatorname{grad}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot e_i.$$

(iii)

$$\operatorname{grad}(f) = \Phi_1(df).$$

(d) divergence

(i) Let F be a vector field on $\mathbb{R}^n$.

(ii) The divergence of F is the function

$$\operatorname{div}(F) = \sum_i \frac{\partial F_i}{\partial x_i}.$$

(iii)

$$\operatorname{div}\left(\Phi_{n-1}(\omega)\right) = \Phi_n(d\omega).$$

(e) curl

(i) $n = 3 \Leftrightarrow n - 1 = 2$

(ii) $\operatorname{curl}(F) = \Phi_2(d(\Phi_1^{-1}(F)))$.