1. Integration

(1) The Riemann integral over a rectangle

  (a) Rectangles

    (i) A subset of $\mathbb{R}$ is connected if and only if it is an interval.

    (ii) A rectangle R in $\mathbb{R}^d$ is a product of closed intervals:

$$R \;=\; [a_1, b_1] \times \cdots \times [a_d, b_d].$$

    (iii) The volume, $\mathrm{Vol}(R)$, of the rectangle is defined to be

$$(b_1 - a_1) \cdot (b_2 - a_2) \cdots (b_d - a_d).$$

  (b) A *finite partition* $\mathcal{P}$ of a rectangle $U \subset \mathbb{R}^d$ is a finite collection of rectangles such that

    (i) $U = \bigcup_{R \in \mathcal{P}} R$, and

    (ii) $\mathrm{interior}(R) \cap \mathrm{interior}(R') = \emptyset$ unless $R = R'$.

  (c) Let $f : \mathbb{R}^d \to \mathbb{R}$.

  (d) Upper sums

    (i) Let $M(f, R) = \sup\{f(x) \mid x \in R\}$.

    (ii) The upper sum of f with respect to P is

$$Uf(P) = \sum_{R \in \mathcal{P}} M(f, R) \cdot \mathrm{Vol}(I).$$

  (e) Lower sums

    (i) Let $m(f, R) = \inf\{f(x) \mid x \in R\}$.

    (ii) The lower sum of f with respect to P is

$$Lf(P) = \sum_{R \in \mathcal{P}} m(f, R) \cdot \mathrm{Vol}(I).$$

  (f) Monotonicity with resepect to refinement:

    (i) We say that $\mathcal{P}'$ is a refinement of $\mathcal{P}$ iff every rectangle in $\mathcal{P}$ is a finite union of rectangles in $\mathcal{P}'$. We write $\mathcal{P}' > \mathcal{P}$.

    (ii) If $\mathcal{P}' > \mathcal{P}$, then

      (A) $Uf(\mathcal{P}') \leq Uf(\mathcal{P})$

      (B) $Lf(\mathcal{P}') \geq Lf(\mathcal{P})$

  (g) *Exercise:* Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of a rectangle U.

    (i) Show that there exists a partiton $\mathcal{S}$ of U such that $\mathcal{S} > \mathcal{P}$ and $\mathcal{S} > \mathcal{Q}$.

    (ii) Show that $Lf(\mathcal{P}) \leq Uf(\mathcal{Q})$.

(h) *Definition:*

(i) We say that f is Riemann integrable over a closed rectangle U if and only if for every $\epsilon > 0$, there exists a finite partition $\mathcal{P}$ so that

$$|Uf(\mathcal{P}) \;-\; Lf(\mathcal{P})| \;<\; \epsilon.$$

(ii) By combining this with the exercise above, we find that

$$-\infty \;<\; \inf_{\mathcal{P}} \; Lf(\mathcal{P}) \;=\; \sup_{\mathcal{P}} \; Uf(\mathcal{P}) \;<\; \infty.$$

(iii) We call this number the Riemann integral of f over U and denote it by

$$\int_U f.$$

(i) *Observation:* If f is not bounded on U, then f is not Riemann integrable over U.

(j) *Theorem:* If f is continuous on a closed rectangle U, then f is Riemann integrable over U.

(k) *Exercise:* Show that the function f defined by

$$f(x) \;=\; \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.

(l) Example:

(i) If x is a rational number, define

$$q(x) \;=\; \inf\,\{q \in \mathbb{N} \mid p/q = x\}.$$

That is, $q(x)$ is the denominator of the reduced form of x.

(ii) For $x \in [0,1]$, define

$$f(x) \;=\; \begin{cases} 1/q(x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(iii) Claim: f is integrable over $[0,1]$.

(iv) First note that $Lf(\mathcal{P}) = 0$ for all partitions $\mathcal{P}$, and hence it suffices to find a partition $\mathcal{P}$ so that $Uf(\mathcal{P}) < \epsilon$.

(v) Given $\epsilon > 0$, choose $q_\epsilon \in \mathbb{N}$ so that $1/q_\epsilon < \epsilon/2$.

(vi) Note that the set

$$F \;=\; \{x \in \mathbb{Q} \cap [0,1] \mid q(x) \leq q_\epsilon\}$$

is finite. (e.g. a discrete subset of a compact space is finite).

(vii) Since F is finite, it is compact, and hence the infimum

$$\delta_F = \;\inf\,\{|x - x'| \;\mid x, x' \in F \text{ with } x \neq x'\,\}$$

is achieved, and particular is positive. (Every point in F is at least δ_F distant from every other point in F.)

(viii) Let n be the number of elements in F.

(ix) For each $x \in F$, let R_x be the closed interval centered at x with

$$\mathrm{Vol}(R_x) \ < \ \min\left\{\frac{\epsilon}{2n}, \delta_F\right\}.$$

(x) Note that

$$\sum_{x\in F} M(f, R_x)\cdot \mathrm{Vol}(R_x) \ \leq \ \frac{\epsilon}{2n}\cdot \sum_{x\in F}\frac{1}{q(x)} \ \leq \ \frac{\epsilon}{2n}\cdot\sum_{x\in F} 1 \ \leq \ \epsilon/2.$$

(xi) The complement of $\bigcup_{x\in F} R_x$ is a finite union of disjoint intervals. Let $R_1, \ldots, R_m$ denote the closures of these intervals.

(xii) Since $R_i \cap R_j = \emptyset$ for $i \neq j$ and each $R_i \subset [0,1]$, we have

$$\sum_{i=1}^{m} \mathrm{Vol}(R_i) \ \leq \ \mathrm{Vol}([0,1]) \ = \ 1.$$

(xiii) Thus,

$$\sum_{i=1}^{m} M(f, R_i)\cdot\mathrm{Vol}(R_i) \ \leq \ \frac{1}{q_\epsilon}\cdot\sum_{i=1}^{m}\mathrm{Vol}(R_i) \ \leq \ \epsilon/2.$$

(xiv) Let

$$\mathcal{P} \ = \ \{R_1, \ldots, R_m\} \cup \{R_x \mid x \in F\}.$$

(2) Content zero and discontinuous functions

(a) Let $A \subset \mathbb{R}^d$.

(b) We say that A has *content zero* if and only if for each $\epsilon > 0$, $\exists$ a finite collection F of rectangles so that

$$A \ \subset \ \bigcup_{R\in F} R,$$

and

$$\sum_{R\in F}\mathrm{Vol}(R) < \epsilon.$$

(c) *Exercise:* Show every countable subset of $\mathbb{R}^d$ has content zero.

(d) Let $\mathrm{Dis}(f)$ denote the set of discontinuities of a function $f : U \to \mathbb{R}$.

(e) *Theorem:* If $\mathrm{Dis}(f)$ has content zero, then f is Riemann integrable on the rectangle U.

(3) Riemann integrability on bounded domains.

(a) Let $\Omega \subset \mathbb{R}^d$ be a bounded subset of $\mathbb{R}^d$.

(b) $\Rightarrow \exists$ rectangle U so that $\Omega \subset U$.

(c) Define the characteristic function $\chi_\Omega : \mathbb{R}^d \to \mathbb{R}$ of Ω by

$$\chi_\Omega(x) \;=\; \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

(d) *Definition:* f is Riemann integrable on $\Omega \Leftrightarrow \chi_\Omega \cdot f$ is integrable over U.

(e) If f integrable over Ω then define

$$\int_\Omega f \;:=\; \int_U \chi_\Omega \cdot f.$$

(f) Integrability of continuous functions on bounded domains.

(i) Let $\partial\Omega = \overline{\Omega} \setminus \text{Interior}(\Omega)$. This is called the *boundary* of Ω.

(ii) *Proposition:* If $\partial\Omega$ has content zero, then each continuous function on Ω is integrable.

(A) $\text{Dis}(\chi_\Omega) \;=\; \partial\Omega$.

(B) If f continuous, then

$$\text{Dis}(\chi_\Omega \cdot f) \;=\; \text{Dis}(\chi_\Omega).$$

(C) Proposition follows from previous theorem.

(4) Integrability in general.

(a) Allow for infinite partitions $\mathcal{P}$ of $\mathbb{R}^d$.

(b) Note that every partition of $\mathbb{R}^d$ is countable.

(c) Riemann sums become infinite sums. How to order rectangles?

(d) Order of summation:

(i) Let $C \subset \mathbb{R}$ be a countable set.

(ii) If $C \subset [0, \infty)$ or $C \subset (-\infty, 0]$, then the sum

$$\sum_{x \in C} x$$

does not depend on an ordering of C.

(e) Definition for functions of one sign

(i) Let f be a nonnegative (or nonpositive) function.

(ii) Upper sums and lower sums are unambiguously defined.

(iii) *Definition:* f is Riemann integrable over $\mathbb{R}^d \Leftrightarrow$ if and only if for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ so that

$$|Uf(\mathcal{P}) \;-\; Lf(\mathcal{P})| \;<\; \epsilon.$$

(f) Given $f : \mathbb{R}^d \to \mathbb{R}$, define the *positive part* of f

$$f^+(x) \;=\; \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \le 0 \end{cases},$$

and define the *negative part* of f

$$f^-(x) \;=\; \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}.$$

(g) Note that $f = f^+ + f^-$, $f^+ \geq 0$ and $f^- \leq 0$.

(h) *Definition:* f is Riemann integrable on $\mathbb{R}^d \Leftrightarrow f^+$ and f^- are integrable. We write

$$\int_{\mathbb{R}^d} f \;=\; \int_{\mathbb{R}^d} f^+ \;+\; \int_{\mathbb{R}^d} f^-.$$

(5) Change of variables formula

(a) Let $\phi : U \to \mathbb{R}^d$ have a continuous derivative.

(b) Let f be an integrable function on $\phi(U)$.

(c) Then

$$\int_{\phi(U)} f \;=\; \int_U f \circ \phi \cdot |\det(d\phi_p)|.$$

(d) Example: $f = 1$ gives volume.

(6) Submanifolds and parametrizations

(a) An *immersion* is a map $\gamma : U \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $d\gamma_p$ has full rank for each $p \in U$.

(b) A parametrized (immersed) n-dimensional sumbmanifold of $\mathbb{R}^m$ is the image of an immersion $\gamma : U \subset \mathbb{R}^n \to \mathbb{R}^m$.

(c) The function γ is called a parametrization of $\gamma(U)$.

(d) Example: Implicit functions give parametrizations of level sets.

(e) If $g : \mathbb{R}^n \to \mathbb{R}^m$ then $x \mapsto (x, g(x))$ is parametrization of the graph of f.

(f) Example: Sphere

(7) Integration on submanifolds with surface measure

(a) Let M be a parametrized submanifold and let $\gamma : U \to M = \gamma(U)$ be a parametrixzation. Define

$$\int_M f \; dA \;=\; \int_U f \circ \gamma \cdot \sqrt{\det(d\gamma^* \circ d\gamma)} \; dx.$$

(b) Change of variables theorem implies that integral is independent of the parametrization.

(c) Special case: γ parameterizes a curve C.

(i) $\gamma : [a, b] \to \mathbb{R}^m$

(ii) $d\gamma_t = \gamma'(t)$ is the 'velocity vector', a vector tangent to curve.

(iii) $\det(d\gamma_t^* \circ d\gamma_t) = |\gamma'(t)|^2$

(iv)

$$\int_C f\ ds \ = \ \int_a^b f \circ \gamma \cdot |\gamma'(t)|\ dt.$$

(d) Special case: parametrization of graph.

(i) Example: $g : \mathbb{R}^2 \to \mathbb{R}$ and $\gamma(x, y) = (x, y, g(x, y))$ then the Jacobian is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

(ii) Thus, $\det(d\gamma^* \circ d\gamma)$ is the determinant of

$$\begin{pmatrix} 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

(iii) Computation gives

$$\det(d\gamma^* \circ d\gamma) \ = \ \sqrt{1 \ + \ \left(\frac{\partial g}{\partial x}\right)^2 \ + \ \left(\frac{\partial g}{\partial y}\right)^2}$$