1. NUMBERS

(1) How do we use numbers?
   (a) To represent solutions to problems.
   (b) Each new type of number concept is generated to express solution to a new problem.
   (c) Better if can be manipulatated to solve problems. (e.g. integers can be added and subtracted.)

(2) Examples:
   (a) the natural numbers, $\mathbb{N}$: solutions to counting problem.
   (b) the rational numbers $\mathbb{Q}^+$: solutions to sharing problem.
   (c) 0 accepted only slowly.
   (d) the integers $\mathbb{Z}$: solutions to accounting problem ('in the red')
   (e) the reals $\mathbb{R}$: solutions to measuring distances, etc.

(3) square root of 2
   (a) $\sqrt{2}$ is the solution to the problem of measuring the length of the diagonal of unit square.
   (b) Pythogorean thm $\Rightarrow \sqrt{2}$ also a solution to $x^2 = 2$.
   (c) Hipparchus proved that there were no rational solutions (500 BCE). (also suggested by *Shulba Sutras* in *Vedas* 800-200 BCE).

(4) divergence of algebra and analysis
   (a) modern algebra: number fields, ideal class group, local rings (Gauss, Kummer, ...)
   (b) modern analysis: normed vector spaces and fixed point theorems (Fréchet, Hilbert, Banach, Brouwer, Schauder, ...)

(5) Babylonian squences
   (a) Babylonians gave rational approximation of $\sqrt{2}$:
      (i) Choose r_0 to be a rational number.
      (ii) Define sequence r_n of rational numbers
      $$r_{n+1} = \frac{r_n}{2} + \frac{1}{r_n}.$$
   (b) Note that $x^2 - 2 = 0$ if and only if
   $$x = \frac{x}{2} + \frac{1}{x}.$$
   (c) We say that x is a *fixed point* of the function $g(x) = \frac{x}{2} + \frac{1}{x}$.
   (d) *Exercise:* If $x > 0$, then $x/2 + 1/x > 1$.

(e) *Exercise:* Show that each Babylonian sequence satisfies
$$|r_{m+1} - r_m| \;\le\; \frac{1}{2} \cdot |r_m - r_{m-1}|.$$

(f) *Idea:* r_m ‘converges’ to a fixed point of g.

(g) But limit can not be rational by Hipparchus.

(6) $\mathbb{R}$ as a completion of $\mathbb{Q}$

(a) Arhimedean norm on $\mathbb{Q}$

(i) Define the *(Archimedean) norm* of a rational number r to be
$$|r| \;=\; \begin{cases} r & \text{if } r < 0 \\ -r & \text{if } 0 < r \\ 0 & \text{if } r = 0 \end{cases}$$

(ii) Triangle inequality: $|r + s| \le |r| + |s|$.

(b) A sequence r_n of rational numbers is *Cauchy* if and only if
$\forall m > 0$, $\exists N$ so that if $i, j > N$ then $|r_i - r_j| < 1/m$.

(c) Example: Babylonian sequences

(i) *Exercise:* Each Babylonian sequence is Cauchy.
(Hint: Use geometric series and prior exercise.)

(ii) If $r_0 \neq r'_0$, then the Babylonian sequences r_n and r'_n are different.

(iii) But one can check that
$$|r_n - r'_n| \;\le\; \frac{1}{2} \cdot |r_{n-1} - r'_{n-1}| \;\le\; \cdots \;\le\; \left(\frac{1}{2}\right)^n \cdot |r_0 - r'_0|.$$

(iv) Thus, $r_n$and r'_n are ‘eventually close’.

(d) Equivalent Cauchy sequences:

(i) Two Cauchy sequences r_n and r'_n are *equivalent* if and only if
$\forall m$, $\exists N$ such that if $n > N$, then $|r_n - r'_n| < 1/m$.

(ii) For example, any two Babylonian sequences are equivalent.

(iii) *Notation:* $r_n \sim r'_n$.

(iv) Note that $\sim$ is an *equivalence relation*, namely we have

(A) $r_n \sim r_n$,

(B) $r_n \sim r'_n \Leftrightarrow r'_n \sim r_n$, and

(C) $r_n \sim r'_n$ and $r'_n \sim r''_n \Rightarrow r_n \sim r''_n$.

(e) Definition of real number

(i) A *real number* is an equivalence class of Cauchy sequences of rational numbers.[1]

[1]Given an equivalence relation $\sim$ on a set, one may partition the set into *classes* such that x and y belong to same class iff $x \sim y$.

(ii) A sequence that belongs an equivalence class is said to *represent* the real number.

(iii) *Notation:* Let $[r_n]$ denote the equivalence that contains r_n.

(iv) The $\sqrt{2}$ is *by definition* the equivalence of Cauchy sequences that contain the Babylonian sequences.

(v) *Exercise:* Find another Cauchy sequence that represents $\sqrt{2}$ that's not Babylonian.

(7) Structure of $\mathbb{R}$

(a) Rationals in the reals.

(i) Given $r \in \mathbb{Q}$, the constant sequence $\{r, r, r, r, \ldots\}$ is Cauchy.

(ii) Define map from $\mathbb{Q}$ to $\mathbb{R}$ by

$$r \mapsto [\{r, r, r, r, \ldots\}].$$

(iii) *Exercise:* This map is injective (one-to-one).

(iv) In practice, we do not distinguish between a rational number and its image in the real numbers.

(b) *Exercise:* Between any two real numbers there exists a rational number (i.e. an image of a rational number).

(c) Addition, subtraction, multipliation, division.

(i) $[r_n] + [s_n] = [r_n + s_n]$

(ii) $[r_n] - [s_n] = [r_n - s_n]$

(iii) $[r_n] \cdot [s_n] = [r_n \cdot s_n]$

(iv) $[r_n]/[s_n] = [r_n/s_n]$

(v) *Exercise:* Show that these operations are well-defined.

(d) Order.

(i) $[r_n] < [r'_n] \Leftrightarrow [r_n] \neq [r'_n]$ and for all m there exists M such that if $n > M$, then we have $r_n < r'_n$.

(ii) *Exercise:* Show that $<$ is well-defined. In particular, what if we change the respesentatives of the equivalence classes?

(iii) *Exercise:* Prove 'trichotomy' for real numbers: Either $x = 0$, $0 < x$, or $x < 0$.

(e) Norm.

(i) *Exercise:* If r_n is a Cauchy sequence of rationals, then $|r_n|$ is a Cauchy squence of rationals.

(ii) Define the *norm* of $[r_n]$ to be $[|r_n|]$.

(iii) If r is rational, then its norm as a real number is the same as its norm as a rational number.

(iv) *Exercise:* Prove the triangle inequality for real numbers: $|x+y| \leq |x| + |y|$.

(8) Convergence in $\mathbb{R}$.

(a) A sequence x_n of real numbers *converges* to a real number x iff $\forall m$, $\exists N$ such that if $n > N$, then $|x - x_n| < 1/m$.

(b) *Exercise:* Let $x = [r_k]$ and $x_n = [r_{n,k}]$. The sequence x_n converges to x iff $\forall m$, $\exists N$ such that if $k, n > N$, then $|r_k - r_{n,k}| < 1/m$.

(c) *Theorem:* If r_n is a Cauchy sequence (of images) of rational numbers, then r_n converges to $[r_n]$.

(i) Given m, choose N so that $i, j > N$ implies $|r_i - r_j| < 1/m$.

(ii) Apply preceding exercise.

(9) The completeness of $\mathbb{R}$.

(a) *Theorem:* Each Cauchy sequence x_n of real numbers converges to a unique real number.

(i) $x_n = [r_{n,k}]$.

(ii) Show that $r_{k,k}$ is Cauchy. ('Diagonalization')

(iii) Apply preceding result on Cauchy sequences of rationals.

(iv) Show that x_n converges to $[r_{k,k}]$.

(b) Converse holds true: If x_n converges to some x, then x_n is Cauchy.

(10) *Theorem:* If $|x_{m+1} - x_m| \leq \frac{1}{2}|x_m - x_{m-1}|$, then x_m is Cauchy.

(a) Pf: Exercise

(b) Can replace $\frac{1}{2}$ by any number $\alpha < 1$.

(11) Supremum and infimum

(a) existence is consequence of order and completeness

(b) A real number u is an *upper bound* for a set $A \subset \mathbb{R}$ iff $u \geq a$ for all $a \in A$.

(c) *Theorem:* If a nonempty set A of real numbers has an upper bound, it has a unique least upper bound.

(d) The least upper bound is called the *supremum* and is denoted $\sup(A)$.

(e) Proof of existence of $\sup(A)$:

(i) Inductive algorithm to define Cauchy sequence that defines to $\sup(A)$ (via limit)

(A) Choose x_0 to be an upper bound and choose y_0 to be a point in A.

(B) Let z_0 be midpoint of x_0 and y_0.

(C) If z_0 is upper bound, then choose $x_1 = z_0$ and $y_1 = y_0$

(D) If z_0 is not upper bound, then choose $x_1 \leq y_0$ to be an upper bound, and choose $y_1 \geq z_0$ to be a point in A.

(E) Let z_1 be midpoint of x_1 and y_1.

(F) Repeat

(ii) Claim: x_n, y_n, and z_n are Cauchy.

(A) Indeed, first note that $|z_{m+1} - z_m| \leq \frac{1}{2}|z_m - z_{m-1}|$, and thus z_m is a Cauchy sequence by exercise above. Let z be the real number represented $\{z_n\}$.

(B) Note that $|z_{m+1} - x_{m+1}| \leq |z_{m+1} - z_m|$ and $|z_{m+1} - y_{m+1}| \leq |z_{m+1} - z_m|$.

(C) *Exercise:* Show that x_m and y_m converge to z.

(iii) Because each y_m belongs to A and each x_m is an upper bound, z is a least upper bound denoted $\sup(A)$.(why?)

(f) *Notation:* If A has no upper bound, then we write $\sup(A) = \infty$. We write $\sup(\emptyset) = -\infty$.

(g) *Exercise:* $A \subset B \Rightarrow \sup(A) \leq \sup(B)$

(h) *Application:* Monotone convergence thm.

(i) If $x_n \leq x_{n+1}$ for all n, then x_n converges to $\sup\{x_n\}$.

(A) $x_n \leq \sup\{x_n\}$ for all n

(B) Given $m > 0$, if $x_n \leq \sup\{x_n\} - 1/m$ for all n then $\sup\{x_n\} - 1/m < \sup\{x_n\}$ is lower bound. Contradiction! Hence there exists M so that $x_M > \sup\{x_n\} - 1/m$.

(C) Monotonicity implies $x_n > \sup\{x_n\} - 1/m$ for all $n > M$.

(ii) Generalizes to: If $f : (a, b) \to \mathbb{R}$ monotone increasing function and $t \in (a, b)$, then $f(t-) = \lim_{r->t_-} f(r)$ exists and equals $\sup\{f(r) : r < t\}$.

(iii) Discontinuities of increasing functions are of 'jump type'

$$f(r) \leq f(t-) \leq f(t) \leq f(t+) \leq f(s)$$

for $r < t < s$.

(12) lim sup and lim inf

(a) Definition of limsup

(i) Let x_n be a sequence of real numbers.

(ii) For each n, let $A_n = \{x_k |\ k \geq n\}$
(sometimes called a 'tail' of the sequence).

(iii) *Exercise:* The sequence x_n is Cauchy if and only if

$$\lim_{n\to\infty} \sup\ \{|x - y|,\ x, y \in A_n\} \ = \ 0.$$

(iv) By completeness, $\sup(A_n)$ exists (in the extended real numbers).

(v) Since $A_{n+1} \subset A_n$, we have $\sup(A_{n+1}) \leq \sup(A_n)$.

(vi) Thus, by monotone convergence theorem

$$\lim_{n\to\infty} \sup\{x_k | \ k \geq n\}$$

exists in the extended real numbers.

(vii) We call this the *limsup* of x_n and denote it by $\limsup_{n\to\infty} x_k$.

(b) Practical criteria: $x = \limsup x_n$ if and only if

(i) $\forall y > x$, the set $\{k \mid x_k > y\}$ is finite, and

(ii) $\forall y < x$, the set $\{k \mid x_k > y\}$ is infinite.

(c) Exercise: *Define liminf and find the analogous 'practical criteria'.*

(d) Application: Root test

(i) $\limsup |a_n|^{\frac{1}{n}} < 1 \Rightarrow \sum a_n$ converges (absolutely).
$\limsup |a_n|^{\frac{1}{n}} > 1 \Rightarrow \sum a_n$ diverges.

(ii) Convergence proof using practical criteria:

(A) Let y lie strictly between $\limsup |a_n|^{\frac{1}{n}}$ and 1.

(B) $\exists M$ so that $n > M \Rightarrow |a_n| < y^n$.

(C) Thus, $\sum_{n>M} |a_n| < \sum_{n>M} y^n < \infty$ since geometric series converge for $y < 1$.

(iii) Divergence proof using practical criteria:

(A) $\forall M$, $\exists n > M$ so that $a_n > 1$

(B) Thus, $\sum_{n>M} |a_n| > \sum_{n>M} 1 = \infty$.

(iv) *Exercise:* Show that

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}.$$

(This implies that the ratio test can be regarded as a consequence of the root test.)

(13) Countability.

(a) Set A is *countable* iff there exists an injection $f : A \to \mathbb{N}$.

(b) *Exercise:* Show that B is countable iff $\exists$ surjection $g : \mathbb{N} \to B$.

(c) *Exercise:* Show that the union of two countable sets is countable. Show that $\mathbb{Z}$ is countable.

(d) *Proposition:* The product of two countable sets is countable.

(i) It suffices to show that $\mathbb{N} \times \mathbb{N}$ is countable (why?).

(ii) For each $n \in \mathbb{N}$, the set $\{(k_1, k_2) \mid k_1 + k_2 = n\}$ has n elements.

(iii) Given $\mathbf{k} = (k_1, k_2) \in \mathbb{N} \times \mathbb{N}$, $\exists$ unique $n_{\mathbf{k}} \in \mathbb{N}$ such that

$$\sum_{n=1}^{n_{\mathbf{k}}} n \;<\; k_1 \;+\; k_2 \;\leq\; \sum_{n=1}^{n_{\mathbf{k}}+1} n.$$

(iv) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by setting $f(k_1, k_2) = n_{\mathbf{k}}$.

(v) Show that f is injective.

(e) $\mathbb{Q}$ is countable.

(i) By above $\mathbb{N} \times \mathbb{Z}$ is countable.

(ii) Define a surjective map from $\mathbb{N} \times \mathbb{Z}$ onto $\mathbb{Q}$.

(f) *Exercise:* If $f : \mathbb{N} \to \mathbb{N}$ is injective, then $\lim_{n\to\infty} f(n) = \infty$.

(g) $\mathbb{R}$ is not a countable set (Cantor's argument)

(i) Suppose to contrary that $\exists$ a surjection $f : \mathbb{N} \to \mathbb{R}$.

(ii) For each n, choose a Cauchy sequence $r_{n,k}$ of rationals that represents $f(n)$.

(iii) Since f is surjective, $\forall j \in \mathbb{N}$, $\exists n(j) \in N$ such that $f(n(j)) = j$.

(iv) If $j \neq j'$, then $n(j) \neq n(j')$.

(v) Exercise $\Rightarrow \lim_{j\to\infty} n(j) = \infty$.

(vi) Show that $r_{n(j),j}$ is Cauchy.

(vii) But $r_{n(j),j} = j$.

(h) *Application:* An increasing function $g : \mathbb{R} \to \mathbb{R}$ has countably many discontinuities.

(i) Let $\mathrm{Dis}(g)$ be the discontinuities of g.

(ii) For $t \in \mathrm{Dis}(g)$, define the open interval

$$I(t) \;=\; \left(\lim_{s\to t^-} g(s),\; \lim_{s\to t^+} g(s)\right).$$

(iii) g increasing $\Rightarrow$ if $t \neq t'$ then $I(t)$ are $I(t')$ disjoint.

(iv) Each interval $I(t)$ contains a rational number $r(t)$.

(v) The map $r : \mathrm{Dis}(g) \to \mathbb{Q}$ is injective.

(i) *Nota Bene:* $\mathbb{R}$ contains countable dense set, namely $\mathbb{Q}$.

2. Additional exercises

(1) Let $g(x) = 1/(1 + 1/x)$. Show that g has a fixed point in the interval $[0, 1]$. What is the value of the fixed point?

(2) Let a_n be the Fibonacci sequence: $a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for all n.

(a) Show that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$.

(b) Show that the partial sums of $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converge.

(3) Let $a_1 \geq 2$, and let $a_{n+1} = \sqrt{2+a_n}$. Show that a_n is monotone and compute the limit.

(4) Let a_n be a sequence of real numbers. Show that for each natural number m we have

$$\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} \leq \limsup_{k\to\infty} |a_{k+m}|^{\frac{1}{k}}.$$